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MULTIPLICATIVE DISTANCE FUNCTIONS

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MULTIPLICATIVE DISTANCE FUNCTIONS

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In memory of Bitey.

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I thank my weekly bridge group for keeping me sane.

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MULTIPLICATIVE DISTANCE FUNCTIONS

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We generalize Mahler's measure to create the class of *multiplicative distance functions* on $\mathbb{C}[x]$. These functions are uniquely determined by their action on the roots of polynomials. We find a simple asymptotic condition that determines which functions on $\mathbb{C}$ are induced by multiplicative distance functions, and use this to give several examples. In particular, we show how Mahler's measure restricted to the set of reciprocal polynomials may be viewed as a multiplicative distance function: the reciprocal Mahler's measure. We then turn to potential theory to demonstrate how new multiplicative distance functions may be created by generalizing Jensen's formula. In so doing we will introduce multiplicative distance functions which measure the complexity of polynomials in $\mathbb{C}[x]$ by comparing the geometry of their roots to compact subsets of $\mathbb{C}$.

Let s be a complex variable, and let N be a positive integer. To every multiplicative distance function Φ we will define an analytic function $F_N(\Phi; s)$ ($H_N(\Phi; s)$ resp.) which encodes information about the range of values Φ takes on degree N polynomials in $\mathbb{R}[x]$ ($\mathbb{C}[x]$ resp.). These functions are analytic in the half plane $\Re(s) > N$. We show that $H_N(s)$ can be represented as the determinant of a Gram matrix in a Hilbert space dependent on s and Φ. This revelation allows us to write $H_N(s)$ as the product of the norms of N vectors in the associated Hilbert space. Several examples are presented. Similarly, when N is even we introduce a skew-symmetric inner product associated to Φ and s and show that $F_N(s)$ can be written as the Pfaffian of an antisymmetric Gram matrix defined from this skew-symmetric inner product. This allows us to write $F_N(s)$ as a product of $N/2$ simpler functions of s. We use this information to compute $F_N(s)$ for the reciprocal Mahler's measure, and in so doing discover that this function is an even rational function of s with rational coefficients and simple poles at small integers.

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Chapter 1

Introduction and Overview

1.1 The Structure of this Thesis

This thesis is separated into two parts: I. Basic Properties of Multiplicative Distance Functions and II. Analytic Functions Associated to Multiplicative Distance Functions. Each part consists of two chapters, which together with this introduction make for a total of five chapters.

This introductory chapter contains a section for each chapter in the thesis. The primary theorems in each Chapter are reported in the corresponding section of the introduction. Many examples and lesser results are absent from the introduction. All proofs are reserved for the main chapters. Additionally, many open problems and research questions surrounding multiplicative distance functions are presented in this introductory chapter.

1.2 Multiplicative Distance Functions

Part I of this thesis is concerned with measures of complexity on polynomials in $\mathbb{C}[x]$ (and $\mathbb{R}[x]$ and $\mathbb{Q}[x]$ by restriction). As such we are interested in functions $\mathbb{C}[x] \to [0, \infty)$ which are sympathetic with the topology and algebraic structure of $\mathbb{C}[x]$. The algebraic structure is that of a graded algebra over $\mathbb{C}$ and we work with the topology generated by all open sets of all finite dimensional subspaces of $\mathbb{C}[x]$.

In Chapter 2 we introduce the concept of a *multiplicative distance function* as a function $\Phi : \mathbb{C}[x] \to [0, \infty)$ such that

1. Φ is continuous,

and for every $k \in \mathbb{C}$ and $f, g \in \mathbb{C}[x]$, Φ is

2. absolutely homogeneous: $\Phi(kf) = |k|\Phi(f)$,
3. positive-definite: $\Phi(f) = 0$ if and only if f is identically zero, and
4. multiplicative: $\Phi(fg) = \Phi(f)\Phi(g)$.

The continuity condition meshes with the topology and additive structure of $\mathbb{C}[x]$, while homogeneity and positive-definiteness mesh with scalar multiplication and multiplicativity is sympathetic with the multiplicative structure of $\mathbb{C}[x]$. The name multiplicative distance function stems from the fact that Φ restricted to any finite dimensional subspace of $\mathbb{C}[x]$ is a distance function in the sense of the geometry of numbers. We will sometimes refer to $\Phi(f)$ as the Φ-distance of f.

It is easily seen that Φ is completely determined by its action on monic linear polynomials in $\mathbb{C}[x]$. That is, if $f(x) = a_N \prod_{n=1}^{N}(x - \alpha_n)$ then there exists a function $\phi : \mathbb{C} \to (0, \infty)$ such that

$$\Phi(f) = |a_N| \prod_{n=1}^{N} \phi(\alpha_n).$$

We call ϕ the *root function* of Φ, and it will play an important role in the theory of multiplicative distance functions. Root functions of multiplicative distance functions are completely characterized by their asymptotic behavior as $\alpha \to \infty$.

Theorem (2.1, p. 34)**.** *Suppose* $\Phi : \mathbb{C}[x] \to [0, \infty)$ *is a multiplicative distance function. Then there exists a continuous function* $\phi : \mathbb{C} \to (0, \infty)$ *with* $\phi(\alpha) \sim |\alpha|$ *so that*

$$\Phi : a_N \prod_{n=1}^{N} (x - \alpha_n) \mapsto |a_N| \prod_{n=1}^{N} \phi(\alpha_n).$$

Theorem (2.2, p.35)**.** *Suppose* $\psi : \mathbb{C} \to (0, \infty)$ *is a continuous function such that* $\psi(\alpha) \sim |\alpha|$ *then the function* $\Psi : \mathbb{C}[x] \to [0, \infty)$ *given by*

$$\Psi : a_N \prod_{n=1}^{N} (x - \alpha_n) \mapsto |a_N| \prod_{n=1}^{N} \psi(\alpha_n)$$

is a multiplicative distance function.

The function $\alpha \mapsto \max\{1, |\alpha|\}$ is (arguably) the simplest root function of a multiplicative distance function. Thus we arrive at our first example of a multiplicative distance function: Mahler's measure. This multiplicative distance function will be denoted by μ. The fact that $\mu(x) = 1$ allows us to naturally extend Mahler's measure to the set of Laurent polynomials $\mathbb{C}[x, 1/x]$. We may use this fact to create new multiplicative distance functions: Let $p(x) \in \mathbb{C}[x]$, and consider μ restricted to $\mathbb{C}[x + p(1/x)]$. Since $\mathbb{C}[x + p(1/x)]$ is canonically isomorphic to $\mathbb{C}[x]$ we may create a new multiplicative distance function μ_p. Specifically, if $f(x) \in \mathbb{C}[x]$ then $\mu_p(f) = \mu(f \circ p)$. The most important non-trivial example of such a multiplicative distance function is Mahler's measure restricted to $\mathbb{C}[x + 1/x]$. We will call this multiplicative distance function *the reciprocal Mahler's measure* and denote it by μ_1. This multiplicative distance function is important since it is essentially Mahler's measure restricted to the set of reciprocal polynomials of even degree. We may create a 'curve' of multiplicative distance functions by defining $\mu_q(f) = \mu(f(x+q/x))$ for a complex number q. We will call μ_q the *q-reciprocal Mahler's* *measure.* The root function of μ_q is given explicitly by

$$\phi_q(\alpha) = \max\left\{1, \left|\frac{\alpha + \sqrt{\alpha^2 - 4q}}{2}\right|\right\} \max\left\{1, \left|\frac{\alpha - \sqrt{\alpha^2 - 4q}}{2}\right|\right\}.$$

So far we have made no mention of *why* we are interested in multiplicative distance functions. An easy answer is that our prototype, Mahler's measure, has been studied extensively and arises in many different areas of mathematics (*e.g.* Diophantine approximation, ergodic theory, algebraic topology). Our interest in the reciprocal Mahler's measure can be explained after introducing the most famous unsolved problem revolving around Mahler's measure.

Unsolved problem (Lehmer's problem, 1933)**.** *Determine if there exists an* $\epsilon > 0$ *such that if* $f(x)$ *is an irreducible, non-cyclotomic polynomial in* $\mathbb{Z}[x]$ *then* $\mu(f) > 1 + \epsilon$.

This problem was first asked in [9] where D. H. Lehmer remarked that

> We have not made an examination of all 10th degree symmetric polynomials, but a rather intensive search has failed to reveal a better polynomial than
>
> $$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1, \quad [\mu] = 1.176280821$$
>
> All efforts to find a better equation of degree 12 and 14 have been unsuccessful.

To date, no better polynomial has been found than Lehmer's degree 10 example. In fact, the supposition that there is a lower bound in Lehmer's problem is often referred to as Lehmer's conjecture. That the lower bound is given by Lehmer's 10th degree polynomial is a stronger form of Lehmer's conjecture.

A partial resolution to Lehmer's problem came in 1971 when C. Smyth reported that if f is an irreducible polynomial which is not reciprocal, then $\mu(f) \geq \mu(x^3 - x - 1) = 1.32\ldots$ [17]. Recall that f is said to be reciprocal if $f(x) = x^{\deg(f)} f(1/x))$. This reduces Lehmer's problem to a problem regarding small values of the reciprocal Mahler's measure.

Unsolved problem (Lehmer's problem redacted)**.** *Determine if there exists an $\epsilon > 0$ such that if $f(x) \in \mathbb{Z}[x]$ is irreducible with not all roots in $[-2, 2]$ then $\mu_1(f) > 1+\epsilon$. In particular is it possible to find such a polynomial f with $\mu_1(f) < \mu_1(x^5 + x^4 - 5x^3 - 5x^2 + 4x + 3)$?*

In 1979 E. Dobrowolski used another method to give a lower bound for the Mahler's measure of an irreducible non-reciprocal polynomial in $\mathbb{Z}[x]$ based on the degree of the polynomial [5]. Specifically, if f is an irreducible non-cyclotomic polynomial of degree N (with N sufficiently large), then there exists $\epsilon > 0$ such that

$$\mu(f) > 1 + (1-\epsilon)\left(\frac{\log\log N}{\log N}\right)^3.$$

Since Dobrowolski's original paper this result has been made effective and the $(1-\epsilon)$ constant has been improved to $9/4$ [18], though the asymptotic term remains the same. Dobrowolski's proof does not make use of Smyth's result — and begs the question whether it is somehow possible to improve the asymptotic term in Dobrowolski's lower bound by restricting our attention to reciprocal polynomials. In the language of the reciprocal Mahler's measure we have the following:

Research question 1. *Can Dobrowolski's method be made to work when μ is replaced with μ_1? If so, is there an improvement in the asymptotic term in Dobrowolski's lower bound?*

We may also ask about analogs of Lehmer's problem for other multiplicative distance functions. For example, the following is a question of R. Rumely [13].

Research question 2 (Rumely)**.** *Does an analog of Lehmer's conjecture hold for some (or almost all)* μ_q *where* $q \in (0,1)$*?*

This thesis in *not* about Lehmer's problem — this connection is only mentioned because it provides a context (one of many) for multiplicative distance functions.

1.3 Potentials and Jensen's Formula

Returning to Mahler's measure as a prototype for multiplicative distance functions, there is an important integral representation of μ. Namely, the Mahler's measure of a Laurent polynomial f can be given by

$$\mu(f) = \exp\left\{ \int_0^1 \log |f(e^{2\pi i\theta})| \, d\theta \right\}.$$

This expression is a consequence of Jensen's formula – and we will refer to it simply as the *Jensen's formula* of μ. The importance of this representation of μ is that it gives us a way to recover the Mahler's measure of f from the coefficients of f as opposed to the roots.

Chapter 3 investigates the existence of analogs of Jensen's formula for multiplicative distance functions other than μ. Namely, if Φ is a multiplicative distance function and there exists a measure ν on $\mathbb{C}$ such that for any $f(x) \in \mathbb{C}[x]$

$$\Phi(f) = \exp\left\{ \int_{\mathbb{C}} \log |f(z)| \, d\nu(z) \right\},$$

then we will call this the Jensen's formula of Φ. A secondary goal of Chapter 3 is the creation of multiplicative distance functions associated to measures and subsets of $\mathbb{C}$ by way of Jensen's formulae.

For some classes of multiplicative distance functions (*e.g.* those with smooth radial root functions) Jensen's formula can be easily recovered from the root function. For other multiplicative distance functions, such as μ_p and μ_q, we need additional tools. These tools come from potential theory, and much of Chapter 3 is a review of those facts about potential theory which allow us to find Jensen's formulae for multiplicative distance functions. If ν is a measure on $\mathbb{C}$ (for simplicity with compact support K), the *potential* of ν is defined to be

$$p_\nu(\alpha) = \exp\left\{ \int_K \log|z-\alpha|\, d\nu(z) \right\}.$$

The potential of ν is subharmonic on $\mathbb{C}$ and $p_\nu(\alpha) \sim |\alpha|$ as $\alpha \to \infty$. Thus, if p_ν is continuous then it is the root function of a multiplicative distance function (which will be denoted P_ν). The *energy* of ν is defined to be

$$I(\nu) = \int_K \int_K \log|z-\alpha|\, d\nu(z)\, d\nu(\alpha).$$

The energy integral is used to distinguish a special probability measure supported on K. It is a non-trivial consequence of potential theory that if $I(\nu) \neq -\infty$ for some probability measure supported on K then there is a unique such measure ν_K which maximizes I. This measure, if it exists, is called the *equilibrium measure* of K, and p_K and P_K will be known as the *equilibrium potential* and *equilibrium multiplicative distance function* of K (resp.). P_K exists for many compact sets K of human interest (*e. g.* connected compact sets which consist of more than one point). More generally if K is a compact subset of $\mathbb{C}$ which is regular with respect to the Dirichlet problem, then P_K exists and is a multiplicative distance function. In this way we find a mechanism for creating new multiplicative distance functions from sufficiently nice compact subsets of $\mathbb{C}$.

It is all fine and good to have a distinguished multiplicative distance function associated to a set K, but we need to know some properties about it before it becomes useful. The most fundamental property of P_K is that it is constant on K. Specifically, it takes the value $\exp(I(\nu_K))$ on K. This value is known as the *capacity* (also *transfinite diameter*) of K. For example, the transfinite diameter of the closed unit disk $\overline{\Delta}$ is 1, and the equilibrium multiplicative distance function is none other than Mahler's measure. The closed interval $[-2, 2]$ on the real axis also has capacity 1, and its equilibrium multiplicative distance function is the reciprocal Mahler's measure. In fact, when $q \in [0, 1]$ the multiplicative distance function μ_q can be written as the equilibrium multiplicative distance function of the region

$$E_q = \left\{ x + iy : \frac{x^2}{(1+q)^2} + \frac{y^2}{(1-q)^2} \leq 1 \right\}.$$

Theorem (3.16, p.67)**.** *Let $f(x) \in \mathbb{C}[x]$ and $q \in [0, 1]$, then $\mu_q(f) = P_{E_q}(f)$.*

In fact this is a consequence of a broader theorem.

Theorem (3.17, p.70)**.** *Let $p(x)$ be a monic polynomial of degree M, and let*

$$F(x) = \frac{p(x)}{x^{M-1}}.$$

If $F'(x)$ does not vanish on $\mathbb{C} \setminus \overline{\Delta}$, then for every $f(x) \in \mathbb{C}[x]$,

$$\mu(f \circ F) = P_K(f),$$

where K is the complement of $F(\mathbb{C} \setminus \overline{\Delta})$.

It is seen from this theorem that multiplicative distance formed from Mahler's measure restricted to certain subalgebras of $\mathbb{C}[x, x^{-1}]$ can be created from equilibrium potentials of lemniscates.

We turn our attention to applications of these potential-theoretic multiplicative distance functions to Diophantine problems. Taking Mahler's measure as our prototype, it is obvious that if $f(x) \in \mathbb{Z}[x]$ has Mahler's measure equal to 1 then f is monic and has all roots on or inside the unit disk. Thus, by a theorem of Kronecker f is in the multiplicative semigroup generated by x and all cyclotomic polynomials. One conclusion is that $\overline{\Delta}$ contains infinitely many complete sets of conjugate algebraic integers. Thus we may ask which other compact sets K contain infinitely many complete sets of conjugate algebraic integers. It is sufficient to consider this question for compact sets K which are closed under complex conjugation. This question was largely settled by M. Fekete and G. Szegö in [7] where they proved that if K is a compact set with transfinite diameter less than 1 then K cannot contain infinitely many complete sets of conjugate algebraic integers, and if K has transfinite diameter equal to 1 then any open neighborhood of K contains infinitely many complete sets of algebraic integers. This second fact implies that if K has transfinite diameter greater than 1 then it contains infinitely many complete sets of conjugate algebraic integers. When the transfinite diameter of K is equal to 1 the Fekete-Szegö Theorem gives us no information.

Determining whether a compact set K of capacity 1 contains infinitely many complete sets of algebraic numbers is equivalent to determining whether or not there are infinitely many irreducible polynomials with P_K equal to 1 (assuming P_K exists). Returning to the Mahler's measure case, the irreducible polynomials with Mahler's measure 1 fall into two categories: those with all roots on the unit circle (cyclotomic), and those without. We may introduce a similar distinction for irreducible polynomials with respect to other equilibrium multiplicative distance functions. Namely, if K is a compact set with positive capacity, then ν_K is supported on the boundary of K. We label irreducible polynomials with all roots in the support of ν_K by K-tomic. For a fixed set K it would be interesting to determine not only if K contains infinitely many complete sets of conjugate algebraic integers, but also whether there are infinitely many K-tomic polynomials and whether there are infinitely many irreducible polynomials with P_K equal to 1 which are not K-tomic. For example, when $q \in [0,1]$, E_q has capacity 1 and is invariant under complex conjugation. It would be interesting to determine for what values of q there are infinitely many E_q-tomic polynomials.

Research question 3. *For what values of q in $[0,1]$ does there exist infinitely many E_q-tomic polynomials? For what values of q does there exist infinitely many non-E_q-tomic irreducible polynomials in $\mathbb{Z}[x]$ with μ_q equal to 1?*

This research question is close in spirit to some work by C. Smyth in which the problem of finding conics with infinitely many complete sets of conjugate algebraic numbers is investigated [16]. In particular, Smyth reports that if an ellipse contains infinitely many complete sets of conjugate algebraic numbers then its foci are either both rational or are conjugate quadratic irrationals. This eliminates some (a.e.) $q \in [0,1]$ for which there are infinitely many E_q-tomic polynomials, but there is still work to be done before the situation is completely resolved.

Related to this question are equidistribution theorems for algebraic numbers of small height. To put this in the context of multiplicative distance functions, given a polynomial $f(x) = a_N \prod_{n=1}^{N}(x - \alpha_n)$ we define the probability measure with point mass at the roots by

$$\nu_f = \frac{1}{N}\sum_{n=1}^{N} \delta_\alpha,$$

where δ_α is the probability measure with unit mass at α. Then Bilu's Equidistribution Theorem states that if $\{f_n\}$ is a sequence of polynomials in $\mathbb{Z}[x]$ such that $\deg(f_n) \to \infty$ and $\mu(f_n) \to 1$, then ν_f converges weakly to normalized Lebesgue measure on the unit circle [1]. Since normalized Lebesgue measure on the circle is exactly the equilibrium measure of $\overline{\Delta}$ this theorem can be interpreted as connecting polynomials with small $P_{\overline{\Delta}}$-distance to the equilibrium measure of $\overline{\Delta}$. In [13] R. Rumely generalized this result to show that if K is a compact subset of $\mathbb{C}$ with capacity 1 which is stable under complex conjugation, and $\{f_n\}$ is a sequence of polynomials in $\mathbb{Z}[x]$ with $\deg(f_n) \to \infty$ and $P_K(f_n) \to 1$ then ν_f converges weakly to the equilibrium measure of K.

This thesis is *not* about K-tomic polynomials, nor polynomials in $f(x)$ with small P_K-distance. These connections are only mentioned to because it ties the concept of a multiplicative distance function to topics of current research interest.

1.4 Complex Moment Functions

Part II of this thesis is concerned with the distribution of values of multiplicative distance functions, with a special emphasis on Mahler's measure and the reciprocal Mahler's measure. The distribution of Mahler's measures was first considered by S-J. Chern and J. Vaaler in [3]. The work here will place their work in a broader context as well as provide additional interpretations of their results.

We will first define the fundamental objects of study in [3]; as we do note that these objects can be defined for more general multiplicative distance functions. Let N be an integer, and consider Mahler's restricted to the set of polynomials of degree at most N. That is, we may view μ as a function on $\mathbb{C}^{N+1}$ by viewing vectors in $\mathbb{C}^{N+1}$ as coefficient vectors of polynomials. As such Mahler's measure satisfies all the axioms of a vector norm except the triangle inequality. Thus, there exists a set akin to the unit ball though this set need not be convex. We will call this set the degree N complex star body of μ and denote it $\mathcal{V}_N$. We analogously define the degree N real star body of μ to be $\mathcal{U}_N = \{\mathbf{a} \in \mathbb{R}^{N+1} : \mu(\mathbf{a}) \leq 1\}$. The star body $\mathcal{U}_N$ is important because the volume (Lebesgue measure) of the dilated star body $T\mathcal{U}_N$ for

$T > 0$ gives an approximation for the number of polynomials with integer coefficients with degree at most N and Mahler's measure at most T. This approximation is especially good in the asymptotic limit $T \to \infty$. By the homogeneity of μ it is easily seen that $\mathrm{vol}(T\mathcal{U}_N) = T^{N+1}\,\mathrm{vol}(\mathcal{U}_N)$, and thus the desired asymptotic estimates are reliant only on the volume of $\mathcal{U}_N$. If we replace polynomials with integer coefficients with polynomials with coefficients in some other lattice in $\mathbb{R}^{N+1}$ then we only need modify the discussion here to include the determinant of the lattice. An analogous discussion can be made for polynomials with coefficients in full rank lattices in $\mathbb{C}^{N+1}$ by replacing the volume of $\mathcal{U}_N$ with the volume of $\mathcal{V}_N$.

Chern and Vaaler make an ingenious use of the Mellin transform in order to determine the values of $\mathrm{vol}(\mathcal{U}_N)$ and $\mathrm{vol}(\mathcal{V}_N)$. This method relies on the *monic* Mahler's measure given by $\widetilde{\mu} : \mathbb{C}^N \to (0,\infty)$ where $\widetilde{\mu}(\mathbf{b})$ is defined to be the Mahler's measure of the monic polynomial whose vector of non-leading coefficients is given by $\mathbf{b}$. Setting λ_N and λ_{2N} as Lebesgue measure on $\mathbb{R}^N$ and $\mathbb{C}^N$ respectively, we define the distribution functions

$$f_N(\xi) = \lambda_N\left\{\mathbf{b} \in \mathbb{R}^N : \widetilde{\mu}(\mathbf{b}) \le \xi\right\} \quad \text{and} \quad h_N(\xi) = \lambda_{2N}\left\{\mathbf{b} \in \mathbb{C}^N : \widetilde{\mu}(\mathbf{b}) \le \xi\right\}.$$

These functions encode information about the range of values of Mahler's measure on the set of monic polynomials of degree N in $\mathbb{R}[x]$ and $\mathbb{C}[x]$ (resp.). The Mellin transforms of these functions are given by

$$\widehat{f_N}(s) = \int_0^\infty \xi^{-s} f_N(\xi)\,\frac{d\xi}{\xi} \quad \text{and} \quad \widehat{h_N}(s) = \int_0^\infty \xi^{-s} h_N(\xi)\,\frac{d\xi}{\xi},$$

where s is a complex variable. When $\Re(s) > N$ these integrals converge to an analytic function. This effectively encodes information about the range of values of Mahler's measure of monic degree N polynomials into analytic functions. These Mellin transforms are important because

$$\mathrm{vol}(\mathcal{U}_N) = \lambda_{N+1}(\mathcal{U}_N) = 2\widehat{f_N}(N+1),$$

and

$$\operatorname{vol}(\mathcal{V}_N) = \lambda_{2N+2}(\mathcal{V}_N) = 2\pi \widehat{h_N}(2N+2).$$

This observation would be useless if it were not possible to evaluate the Mellin transforms. However, after a change of variables (in the Lebesgue-Stieltges sense) Chern and Vaaler are able to write $\widehat{f_N}(s)$ and $\widehat{h_N}$ as

$$\widehat{f_N}(s) = \frac{1}{s}\int_{\mathbb{R}^N} \widetilde{\mu}(\mathbf{b})^{-s}\, d\lambda_N(\mathbf{b}) =: \frac{1}{s}F_N(s),$$

and

$$\widehat{h_N}(2s) = \frac{1}{2s}\int_{\mathbb{C}^N} \widetilde{\mu}(\mathbf{b})^{-2s}\, d\lambda_{2N}(\mathbf{b}) =: \frac{1}{2s}H_N(s).$$

The functions $F_N(s)$ and $H_N(s)$ converge when $\Re(s) > N$, and are termed the real and complex *moment functions* of μ. After a change of variables $F_N(s)$ and $H_N(s)$ can be written as integrals over root vectors of polynomials which is the first step in their evaluation.

Chern and Vaaler, after a dispiriting (but admirable) foray into rational function identities were able to demonstrate that both $F_N(s)$ and $H_N(s)$ analytically continue to rational functions with simple poles at integers and high multiplicity zeros at the origin.

Theorem (S.J. Chern, J. Vaaler)**.** *Let J be the integer part of $N/2$. Then,*

$$H_N(s) = \frac{\pi^n}{N!}\prod_{n=1}^{N}\frac{s}{s-n},$$

and

$$F_N(s) = \left\{2^N \prod_{j=1}^{J}\left(\frac{2j}{2j+1}\right)^{N-2j}\right\}\prod_{j=0}^{J-1}\frac{s}{s-(N-2j)}.$$

An immediate consequence is that the volume of $\mathcal{U}_N$ is a rational number! Moreover, since $F_N(s)$ and $H_N(s)$ are essentially the Mellin transforms of f_N and h_N explicit formulae for f_N and h_N, can be recovered using the

Mellin inversion formula. In this manner Chern and Vaaler showed that f_N were both polynomials of degree N, and moreover f_N has rational coefficients. The Mellin transform technique, originally introduced to find the volume of star bodies, yielded many surprising and unlooked-for results.

The mechanism by which the volumes of $\mathcal{U}_N$ and $\mathcal{V}_N$ are represented as special values of Mellin transforms is applicable to more general multiplicative distance functions. And thus, given a multiplicative distance function we may introduce the star bodies $\mathcal{U}_N(\Phi)$ and $\mathcal{V}_N(\Phi)$, the distribution functions $f_N(\Phi;\xi)$ and $h_N(\Phi;\xi)$ and the moment functions $F_N(\Phi;s)$ and $H_N(\Phi;s)$. When generalizing Chern and Vaaler's method to other multiplicative distance functions, the exact mechanism by which $F_N(\mu;s)$ and $H_N(\mu;s)$ were written as rational functions seemed specific to Mahler's measure. However, it was hoped that a device could be introduced which would show how these moment functions could be written as products of simpler functions. The first step in this regard was the evaluation of H_N for the reciprocal Mahler's measure. As we shall see, the evaluation of complex moment functions is much simpler than the evaluation of real moment functions.

It was demonstrated in [15] that $H_N(\mu_1;s)$ has an analytic continuation to an even or odd rational function of s.

Theorem (S-, [15])**.**

$$H_N(\mu_1;s) = (2\pi)^N \prod_{n=1}^{N} \frac{s}{s^2-n^2}.$$

This function encodes information about the range of values of Mahler's measure on the set of reciprocal Laurent polynomials in $\mathbb{C}[x,x^{-1}]$ of degree at most N. This being said, an interesting corollary of this theorem is that the complex distribution function $h_N(\mu_1;\xi)$ is itself a reciprocal Laurent polynomial of degree N.

We will see that this result may be extended to include explicit formulae for $H_N(\mu_q)$ when $q \in [0, \infty)$].

Theorem (4.7, p.88)**.** *If $q \in [0, 1]$, then $H_N(\mu_q; s)$ analytically continues to the rational function of s given by*

$$H_N(\mu_q; s) = \frac{\pi^N s^N}{N!} \prod_{n=1}^{N} \frac{(1 - q^{2n})s + (1 + q^{2n})n}{s^2 - n^2}.$$

If $q \in (1, \infty)$ then $H_N(\mu_q; s)$ analytically continues to the meromorphic function of s given by

$$H_N(\mu_q; s) = \frac{q^{-2sN} \pi^N s^N}{N!} \prod_{n=1}^{N} \frac{(-1 + q^{2n})s + (1 + q^{2n})n}{s^2 - n^2}.$$

When $q \in [0, 1]$ we see that $H_N(\mu_q; s)$ is polynomial in q and rational in s. Moreover, since Mahler's measure is μ_0 and the reciprocal Mahler's measure is μ_1, we have explicit formulae for a *path* of complex moment functions connecting $H_N(\mu; s)$ and $H_N(\mu_1; s)$. Figure 1.1 shows a plot of the location of the poles and zeros of $H_6(\mu_q; s)$ as q varies from 0 to 1. In particular notice how the non-zero zeros conspire to cancel the poles at negative integers when $q = 0$, and how the non-zero zeros head toward $-\infty$ as $q \to 1$. The presence of

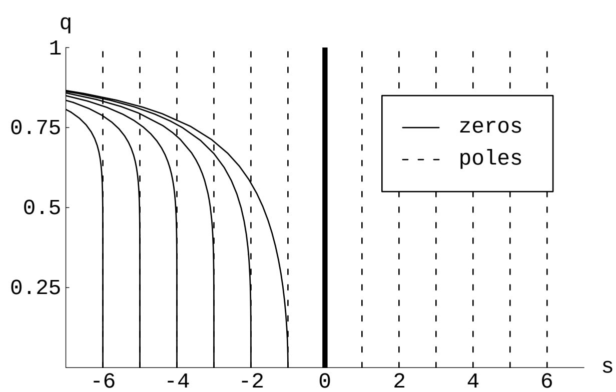

Figure 1.1: The location of the poles and zeros of $H_6(\mu_q; s)$

a product formulation for these examples was welcome news, and suggested a method for finding such a product formulation for complex moment functions of more general multiplicative distance functions.

The key to the discovery of a product formulation of $H_N(\mu_q; s)$ was the discovery that this moment function can be written as the determinant of a matrix in a certain Hilbert space associated to μ_q. Moreover the determinant which arises is of a special kind of matrix known as a Gram matrix – a fact which allows us to interpret $H_N(\mu_q; s)$ as the volume of a parallelepiped in the associated Hilbert space.

In order to do this we define the complex measure $\nu = \nu(\Phi)$ on $\mathbb{C}$ defined by $d\nu(\alpha) = \phi(\alpha)^{-s}\phi(\overline{\alpha})^{-s}\, d\lambda_2(\alpha)$. We view s as a parameter to be chosen later. Then, $L^2(\nu)$ is a Hilbert space equipped with the inner product

$$\langle f|g\rangle = \int_{\mathbb{C}} \phi(\alpha)^{-2s} f(\alpha)\overline{g(\alpha)}\, d\lambda_2(\alpha) \qquad f, g \in L^2(\nu),$$

and norm given by $\mathfrak{N}(f)^2 = \mathfrak{N}(f; s)^2 := \langle f|f\rangle$. When $\Re(s) > N$ then any polynomial in $\mathbb{C}[x]$ with degree less than N is in $L^2(\nu)$.

Now, let $Q = \{Q_n(\alpha) : n = 1, 2, \ldots, N\}$ be a set monic polynomials in $\mathbb{C}[x]$ with $\deg(Q_n) = n - 1$. We will call such a set a *complete* family of polynomials. Each polynomial Q_n is in $L^2(\nu)$ and Q spans a parallelepiped in this Hilbert space. The Gram matrix of Q is defined to be the $N \times N$ matrix, whose j, k entry is given by $\langle Q_j|Q_k\rangle$. This is a symmetric matrix whose j, k entry is dependent on Q_j, Q_k, Φ and s. The determinant of this matrix can be interpreted as the *volume* of the parallelepiped spanned by Q in the Hilbert space $L^2(\nu)$.

Theorem (4.3, p.85)**.** *Let Q be any complete family of monic polynomials. Then*

$$H_N(s) = \det(W_Q).$$

This theorem is exactly the answer to our question – by choosing a family of monic polynomials which is orthogonal with respect to our inner product we will have $H_N(\Phi; s)$ written as the determinant of a diagonal matrix, and the diagonal entries in the matrix will each be functions of s.

Corollary (4.4, p.85)**.** *Let $\Re(s) > N$, and let Q be a complete family of monic polynomials with*

$$\langle Q_j | Q_k \rangle = \delta_{jk}\, \mathfrak{N}(Q_j; s)^2,$$

where $\delta_{jk} = 1$ if $j = k$ and is 0 otherwise. Then,

$$H_N(s) = \prod_{n=1}^{N} \mathfrak{N}(Q_n; s)^2.$$

It is easy to verify that the family of monic orthogonal polynomials associated to μ is simply $\{1, x, \ldots, x^{N-1}\}$ – this fact explains why the naïve computation of $H_N(\mu; s)$ is easy compared to the computation of complex moment functions of other multiplicative distance functions. In fact if Φ is any multiplicative distance function with radial root function, then $\{1, x, x^2, \ldots, x^{N-1}\}$ is a family of orthogonal polynomials with respect to the corresponding inner product. It would be interesting to determine families of orthogonal polynomials for other multiplicative distance functions.

Research question 4. *What are the families of monic orthogonal polynomials with respect to the inner products associated to μ_q when $q \in (0, 1]$?*

1.5 Real Moment Functions

Chern and Vaaler's method of evaluation of $F_N(\mu; s)$ is much more complicated than that of the complex moment functions. The additional complexity stems from the fact that $\mathbb{R}$ is not algebraically complete. In the complex case a single change of variables allows the complex moment function, originally defined as an integral over coefficient vectors of polynomials, to be represented as an integral over the root vectors of polynomials. For real moment functions the space of coefficient vectors of real polynomials must first be partitioned into regions depending on the number of real and complex conjugate pairs of roots. The next step is to employ a separate change of variables for each region in this partition. We are left with a combinatorial formula involving a sum of integrals over $\mathbb{R}^L \times \mathbb{C}^M$ where $L + 2M = N$. To be concrete, we introduce the change of variables $E_{L,M} : \mathbb{R}^L \times \mathbb{C}^M \to \mathbb{R}^N$ given by $E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathbf{a}$ where

$$x^N + \sum_{n=0}^{N-1} a_n x^n = \prod_{\ell=1}^{L} (x - \alpha_\ell) \prod_{m=1}^{M} (x - \beta_m)(x - \overline{\beta}_m).$$

Then, since the degree of $E_{L,M}$ is $2^M M! L!$, we can write $F_N(\Phi; s)$ as

$$\sum_{L+2M=N} \frac{1}{2^M L! M!} \int\limits_{\mathbb{R}^L \times \mathbb{C}^M} \widetilde{\Phi}\left(E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta})\right)^{-s} |\operatorname{Jac}(E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta})|\, d(\lambda_L \times \lambda_{2M})(\boldsymbol{\alpha}, \boldsymbol{\beta}).$$

From this formula we can see the second major hurdle in the evaluation of $F_N(\Phi; s)$: the evaluation of $\operatorname{Jac}(E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta}))$, and determining the regions on which it is positive and negative. It turns out that $\operatorname{Jac}(E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ can be expressed in terms of a Vandermonde determinant. Chern and Vaaler expand this determinant as a sum over the symmetric group and use the well-known relationship between the Vandermonde and the discriminant to write $F_N(\Phi; s)$ as

$$\sum_{L+2M=N} \frac{(-i)^M}{L! M!} \sum_{\tau \in S_N} \operatorname{sgn}(\tau) \int\limits_{\mathbb{R}^L} \prod_{l=1}^{L} \phi(\alpha_l)^{-s} \alpha_l^{\tau(l)-1} \prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j)\, d\lambda_L(\boldsymbol{\alpha})$$
$$\times \prod_{m=1}^{M} \int\limits_{\mathbb{C}} \phi(\overline{\beta})^{-s} \phi(\beta)^{-s} (\overline{\beta})^{\tau(L+2m-1)-1} \beta^{\tau(L+2m)-1} \operatorname{sgn}(\Im(\beta))\, d\lambda_2(\beta).$$

When $\phi(\alpha) = \max\{1, |\alpha|\}$ the integrals in this expression are elementary, and evaluate to rational functions of s with rational coefficients. In fact, when ϕ is the root function of the reciprocal Mahler's measure it can be show that each of the integrals in this expression evaluates to an even rational function of s with rational coefficients. In order to give their simple product formulation of $F_N(\mu; a)$ delve into a number of combinatorial rational function identities. Unfortunately the rational function identities they employed are specific to Mahler's measure, and cannot be generalized to more general multiplicative distance functions.

The most important result presented in this thesis is a method for evaluating $F_N(\Phi; s)$ for general multiplicative distance functions. Using this method we will see how and why the product formulation arises for $F_N(\mu; s)$ and see why we expect similar products to appear for other multiplicative distance functions. Moreover we will see a strong analogy between the evaluation of $H_N(\Phi; s)$ and that of $F_N(\Phi; s)$. In so doing we will avoid the necessity of decomposing space based on the number of real and complex roots, and the use of the change of variables $E_{L,M}$ (with its pesky Jacobian).

The product formulation of $F_N(\Phi; s)$ for arbitrary multiplicative distance functions is dependent on two *skew-symmetric* inner products dependent on Φ. Given functions $f, g : \mathbb{C} \to \mathbb{C}$ which are real on the real axis, define the skew-symmetric inner products by

$$\langle f, g\rangle_{\mathbb{R}} := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x)^{-s}\phi(y)^{-s} f(x) g(y) \operatorname{sgn}(y - x)\, dx\, dy,$$

and

$$\langle f, g\rangle_{\mathbb{C}} := -2i \int_{\mathbb{C}} \phi(\beta)^{-s}\phi(\overline{\beta})^{-s}\overline{f(\beta)} g(\beta) \operatorname{sgn}(\Im(\beta))\, d\lambda_2(\beta).$$

In fact, each s creates a different pair of skew-symmetric inner products, though for the moment we will think of s simply as a complex parameter. The *skew-symmetric* moniker is due to the fact that $\langle g, f\rangle_{\mathbb{R}} = -\langle f, g\rangle_{\mathbb{R}}$ (and similarly for $\langle \cdot, \cdot \rangle_{\mathbb{C}}$). We use these two skew-symmetric inner products to create a new skew-symmetric inner product given by

$$\langle f, g \rangle = \langle f, g \rangle_{\mathbb{R}} + \langle f, g \rangle_{\mathbb{C}}.$$

Of course, $\langle \cdot, \cdot \rangle$ is distinct from the inner product $\langle \cdot | \cdot \rangle$ we introduced earlier. As before we let $\{Q_n(x) : n = 1, \ldots, N\}$ be any family of monic polynomials in $\mathbb{R}[x]$ with $\deg(Q_n) = n - 1$, and we define the $N \times N$ matrix U_Q by $U_Q(j,k) = \langle Q_j, Q_k \rangle$. This matrix is antisymmetric.

An important invariant of antisymmetric matrices is the Pfaffian. Like the determinant, the Pfaffian can be expressed as a sum over the symmetric group. For instance, when $N = 2J$, the Pfaffian of U_Q is defined to be

$$\mathrm{Pf}(U_Q) := \frac{1}{2^J J!} \sum_{\tau \in S_N} \mathrm{sgn}(\tau) \prod_{j=1}^{J} U_Q(\tau(2j-1), \tau(2j)).$$

The Pfaffian is closely related to the determinant — for instance, $\det(U_Q) = \mathrm{Pf}(U_Q)^2$.

Much like the $H_N(\Phi; s)$ can be expressed as the determinant of W_Q, $F_N(\Phi; s)$ can be expressed at the Pfaffian of U_Q.

Theorem (5.2, p.101)**.** *Let N be an even integer, and let Q be any complete family of monic polynomials in $\mathbb{R}[x]$. Then,*

$$F_N(s) = \mathrm{Pf}(U_Q).$$

As with complex moment functions, a smart choice of Q will allow us to find a product formulation for $F_N(s)$. Specifically, we would like to find a complete family of polynomials Q in $\mathbb{R}[x]$ such that for $1 \leq j, k \leq J$,

$$\langle Q_{2k-1}, Q_{2j} \rangle = -\langle Q_{2j}, Q_{2k-1} \rangle = r_j \delta_{kj},$$

and

$$\langle Q_{2j}, Q_{2k} \rangle = \langle Q_{2j-1}, Q_{2k-1} \rangle = 0.$$

In this situation we will say that Q is a complete *skew-orthogonal* family of polynomials. Notice that, since $\langle \cdot, \cdot \rangle$ is dependent on s, then r_j too is a function of s and we will write $r_j = r_j(s)$.

Corollary (5.4, p.102)**.** *Let $N = 2J$ and suppose $\Re(s) > N$. Furthermore, let Q be a complete skew-orthogonal family of monic polynomials in $\mathbb{R}[x]$. Then,*

$$F_N(s) = \prod_{j=1}^{J} r_j(s).$$

For certain multiplicative distance functions, $F_N(\Phi; s)$ can be written not only as the Pfaffian of a $2J \times 2J$ matrix, but can also be written as the determinant of a $J \times J$ matrix.

Theorem (5.3, p.101)**.** *Let $N = 2J$, and let Q be any complete family of monic polynomials in $\mathbb{R}[x]$ such that Q_n is even when $n-1$ is even, and Q_n is odd when $n-1$ is odd. Further suppose the root function of Φ satisfies $\phi(-\beta) = \phi(\beta)$ and $\phi(\overline{\beta}) = \phi(\beta)$ for every $\beta \in \mathbb{C}$. Then,*

$$F_N(s) = \det(A_Q)$$

where A_Q is the $J \times J$ matrix whose j, k entry is given by

$$A_Q(j,k) = U_Q(2j-1, 2k).$$

This theorem is valuable since the determinant is a more familiar entity than the Pfaffian. Moreover many of the examples of multiplicative distance functions presented in this thesis satisfy the hypotheses of this theorem (*e.g.* μ_q for $q > 1$). We will use this theorem to recover the formulation of $H_N(\mu; s)$ given by Chern and Vaaler when N is even. We will also use this theorem to present a simple product formula for $F_N(\mu_1; s)$ when N is even.

Theorem (5.7, p.106)**.** *Let $N = 2J$. Then,*

$$F_N(\mu_1; s) = v_N \prod_{j=0}^{J-1} \frac{s^2}{s^2 - (N-2j)^2}, \quad \textit{where} \quad v_N = \frac{2^N}{N!} \prod_{n=1}^{N} \left(\frac{2n}{2n-1}\right)^{N+1-n}.$$

Notice the similarities between this expression and Chern and Vaaler's formulation of $F_N(\mu; s)$ given on page 13.

1.6 Open Questions About Moment Functions

Many questions remain regarding real moment functions. Foremost perhaps, is how does one find product formulations for $F_N(s)$ when N is odd?

Research question 5. *Can $F_N(s)$ be realized as the Pfaffian of a matrix when N is odd?*

If N is odd, then the Pfaffian of an $N \times N$ matrix is not even defined. In spite of this difficulty, I think it is possible to create an $N+1 \times N+1$ antisymmetric matrix such that $F_N(s)$ can be realized as the Pfaffian of this matrix.

More examples of explicit formulae for real moment functions would be valuable for determining how the analytic properties of $F_N(\Phi; s)$ are related to the root function of Φ.

Research question 6. *What is the product formulation for $F_N(\mu_q; s)$ when $0 < q < 1$?*

It would be interesting to compute the skew-orthogonal polynomials for μ_q when $0 \leq q \leq 1$.

Research question 7. *What are the families of monic skew-orthogonal polynomials with respect to the skew-symmetric inner products associated to μ_q*

when $q \in [0,1]$? How are these families related to the corresponding families of orthogonal polynomials with respect to the inner products associated to these multiplicative distance functions?

It would also be interesting to determine for which multiplicative distance functions $F_N(s)$ and $H_N(s)$ have analytic continuation beyond $\Re(s) > N$.

Research question 8. *What must be true of Φ and ϕ so that $F_N(\Phi; s)$ and $H_N(\Phi; s)$ have an analytic continuation beyond $\Re(s) > N$. If $H_N(\Phi; s)$ has an analytic continuation to a larger domain, does this imply that $F_N(\Phi; s)$ too has an analytic continuation to a larger domain?*

The fact that $F_N(\mu; s)$ and $F_N(\mu_1; s)$ have analytic continuation to rational functions of s begs the question:

Research question 9. *What must be true of Φ and ϕ so that $F_N(\Phi; s)$ and/or $H_N(\Phi; s)$ have analytic continuation to a rational function of s? In this situation, when are the poles at integers? When are the poles simple? When are the zeros at 0? Can the total number of zeros and poles be determined from Φ?*

I hazard to speculate that if Φ is a multiplicative distance function formed as in Theorem 3.17 — that is $\Phi(f) = \mu(f \circ F)$ for some appropriate Laurent polynomial F — then $F_N(\Phi; s)$ has an analytic continuation to a rational function of s.

Conjecture 1. *Let P_K be a multiplicative distance function where K and $p(x)$ are given as in Theorem 3.17. Then,*

1. *$F_N(P_K; s)$ and $H_N(P_K; s)$ have analytic continuations to rational functions of s.*

2. *If $p(x) \in \mathbb{Q}[x]$ then $F_N(P_K; s)$ is in $\mathbb{Q}(s)$, and $H_N(P_K; s)$ is π^N times a rational function in $\mathbb{Q}(s)$.*

3. *$F_N(P_K; s)$ and $H_N(P_K; s)$ have poles at integers $\leq N$.*

Assuming this conjecture is correct, it may show that $F_N(\Phi; s)$ has poles only at integers for a larger class of multiplicative distance functions. Namely, suppose K is a simply connected domain of capacity 1. Then there is a conformal map from the complement of the closed unit disk to the complement of K which takes ∞ to ∞. The Taylor series of this map about the point ∞ will look something like $x + \rho(1/x)$ where $\rho(t)$ is some power series defined in the closed unit disk. Then, by truncating the Taylor series we may find a family of multiplicative distance functions which satisfy the hypothesis of the conjecture. If the conjecture holds, and the the moment functions of these multiplicative distance function converge in some sense to $F_N(P_K; s)$, then we may find that $F_N(P_K; s)$ has poles only at integers $\leq N$.

Part I

Basic Properties of Multiplicative Distance Functions

Chapter 2

Multiplicative Distance Functions

In this chapter we will formalize the notion of a multiplicative distance functions. Multiplicative distance functions are generalizations to $\mathbb{C}[x]$ of distance functions in the theory of the geometry of numbers. We take our inspiration for multiplicative distance functions from Mahler's measure. And, much as Mahler's measure is determined by the function $\alpha \mapsto \max\{1, |\alpha|\}$, we shall see that multiplicative distance functions are determined by their behavior on the roots of polynomials. We shall give a characterization of multiplicative distance functions by the asymptotic behavior of their *root function*. And, in a certain sense, we shall see that the function $\alpha \mapsto \max\{1, |\alpha|\}$ is the simplest root function, keeping Mahler's measure at the center of this theory. From this characterization of multiplicative distance functions we will introduce several other examples of multiplicative distance functions. In particular the reciprocal Mahler's measure, a central example in this thesis, will be formally introduced.

2.1 Distance Functions on $\mathbb{C}[x]$

A function $\Phi : \mathbb{C}[x] \to [0, \infty)$ on $\mathbb{C}[x]$ is called a *distance function* if,

1. Φ is continuous,

and for all f in $\mathbb{C}[x]$ and all $k \in \mathbb{C}$, it is

2. absolutely homogeneous: $\Phi(kf) = |k|\Phi(f)$, and

3. positive-definite: $\Phi(f) = 0$ if and only if f is identically zero.

We shall call $\Phi(f)$ the distance of f from the origin, or simply the *distance* of f. If confusion may arise we may also use the terminology Φ-distance for $\Phi(f)$. Φ is *natural* if $\Phi(x^m f) = \Phi(f)$ for each positive integer m. Natural distance functions can be extended to the set of Laurent polynomials, which we will sometimes do without further comment.

The notion of a distance function on $\mathbb{C}[x]$ is an extension of the notion of a distance function on a finite dimensional vector space: If W is a finite dimensional vector space over either $\mathbb{R}$ or $\mathbb{C}$, then a function $W \to \mathbb{R}$ that is non-negative, absolutely homogeneous and positive-definite is known as a distance function. Distance functions are fundamental objects in the study of the geometry of numbers; see [2] for an in-depth treatment. A distance function on $\mathbb{C}[x]$ can be regarded as a family of distance functions $\{\Phi_N : \mathbb{C}^{N+1} \to [0,\infty)\}$ such that for each $N > 0$ and $\mathbf{a} \in \mathbb{C}^N$,

$$\Phi_N(a_0, \ldots, a_{N-1}, 0) = \Phi_{N-1}(\mathbf{a}).$$

Before continuing, a discussion of the continuity of distance functions on $\mathbb{C}[x]$ is in order. What does it mean for a function $\Phi : \mathbb{C}[x] \to [0,\infty)$ to be continuous? Viewing Φ as a family of distance functions $\{\Phi_N : N \in \mathbb{N}\}$ as in the previous paragraph, then we will say Φ is continuous if Φ_N is continuous for every $N > 0$. Some distance functions we will work with will be continuous with respect to the stronger topology induced by uniform convergence on compact subsets of $\mathbb{C}$ (viewing polynomials as maps on $\mathbb{C}$). We will call this topology the *strong* topology on $\mathbb{C}[x]$. For most purposes the weaker topology induced by convergence on finite dimensional subspaces of $\mathbb{C}[x]$ is sufficient.

Distance functions measure the complexity of polynomials based on their coefficient vectors. Examples of distance functions can be constructed from vector norms, though distance functions are more general since they need not satisfy the triangle inequality. The following two examples give classes of distance functions induced by vector norms.

Example 2.1.1 (the p-norm on coefficients)**.** Given $\mathbf{a}(x)$ of degree N, and positive real parameter p, define

$$|\mathbf{a}|_p = \left\{ \sum_{n=0}^{N} |a_n|^p \right\}^{1/p}.$$

That is, $|\mathbf{a}|_p$ is the usual p-norm on $\mathbb{C}^{N+1}$. It is easy to verify that $|\cdot|_p$ is a distance function. In spite of the name, $|\cdot|_p$ is not a vector norm if $p < 1$ (though it remains a distance function in this situation).

$|\mathbf{a}|_1$ is known as the length of $\mathbf{a}\langle x\rangle$. The function

$$|\mathbf{a}|_\infty = \lim_{p\to\infty} |\mathbf{a}|_p = \max\{|a_0|, \ldots, |a_N|\}$$

is known as the height of $\mathbf{a}(x)$.

Example 2.1.2 (the p-norm on the unit circle)**.** Given a polynomial $\mathbf{a}(x)$, and positive real parameter p, define

$$\|\mathbf{a}\|_p = \left\{ \int_0^1 \left|\mathbf{a}(e^{2\pi i\theta})\right|^p \, d\theta \right\}^{1/p}.$$

It is easy to verify that $\|\cdot\|_p$ is a distance function function. When $p = 2$, by Parseval's formula we find that $|\cdot|_2 = \|\cdot\|_2$.

Both $|\cdot|_p$ and $\|\cdot\|_p$ are natural distance functions when $p > 0$.

A distance function Φ is called multiplicative if $\Phi(fg) = \Phi(f)\Phi(g)$ for all f, g in $\mathbb{C}[x]$. Multiplicative distance functions measure the complexity of polynomials based on their coefficient vectors *and* their roots. It is this duality which makes multiplicative distance functions important objects in the study of complexity of polynomials. Viewing a multiplicative distance function as a function on root vectors provides different information than its action on coefficient vectors and the change of variables from root vectors to coefficient vectors plays a central role in the subject. And, while it is in general a difficult problem to recover the roots of a polynomial from the coefficients, much can be learned about the value of multiplicative distance functions on "average" (or random) polynomials.

The most important example of a multiplicative distance function is Mahler's measure.

Example 2.1.3. Given a polynomial $f \in \mathbb{C}[x]$, the Mahler's measure of f is defined to be the quantity

$$\mu(f) = \lim_{p \to 0} \|f\|_p = \exp\left\{ \int_0^1 \log|f(e^{2\pi i\theta})|\, d\theta \right\}. \tag{2.1}$$

It is easy to verify that the function $\mu : \mathbb{C}[x] \to [0, \infty)$ is positive definite, absolutely homogeneous and multiplicative. It was discovered to be continuous by K. Mahler in 1961 [10]. A proof of the continuity of general multiplicative distance functions will be presented later in this chapter. Assuming continuity, we conclude that μ is indeed a multiplicative distance function.

If f is a polynomial of degree N that factors over the complex numbers as $f(x) = a_N \prod_{n=1}^{N}(x - \alpha_n)$, then by Jensen's formula,

$$\mu(f) = |a_N| \prod_{n=1}^{N} \max\{1, |\alpha_n|\}. \tag{2.2}$$

This formulation of Mahler's measure will provide a model for creating other examples of multiplicative distance functions.

Given a distance function Φ on $\mathbb{C}[x]$, one topic of interest to number theorists is the cardinality of the set of integer polynomials of bounded degree and distance bounded by a constant. This number is always finite. More generally, if Λ is a discrete lattice in $\mathbb{C}^{N+1}$ then the cardinality of the set of elements of Λ with distance bounded by a constant is finite. To see this, let

$$\mathcal{V}_N = \mathcal{V}_N(\Phi) = \{\mathbf{v} \in \mathbb{C}^{N+1} : \Phi(\mathbf{v}) \leq 1\}$$

We call $\mathcal{V}_N(\Phi)$ the *degree N star body of* Φ. $\mathcal{V}_N$ is analogous to the unit ball of a vector norm — though it may not be convex since Φ need not satisfy the triangle inequality. By the homogeneity of Φ, for each $\mathbf{w} \in \mathbb{C}^{N+1}$, the vector $\mathbf{w}/\Phi(\mathbf{w})$ is on the boundary of $\mathcal{V}_N$. It follows that $\mathcal{V}_N$ is a closed bounded subset of $\mathbb{C}^{N+1}$. Now, the set of coefficient vectors of degree N and distance bounded by positive constant T is the dilated star body $T\mathcal{V}_N$. As before, λ_{2N} is Lebesgue measure on Borel subsets of $\mathbb{C}^N$, and the number $\lambda_{2N+2}(\mathcal{V}_N)$ will be called the *volume* of $\mathcal{V}_N$. By the homogeneity of Φ we have

$$\lambda_{2N+2}(T\mathcal{V}_N) = T^{2N+2}\lambda_{2N+2}(\mathcal{V}_N).$$

Then the set of elements of elements in Λ with distance bounded by T is given by

$$\{\mathbf{v} \in \Lambda : \Phi(\mathbf{v}) \leq T\} = T\mathcal{V}_N \cap \Lambda.$$

Since $\mathcal{V}_N$ is bounded it follows that the cardinality of the set of polynomials in $\Lambda(x)$ with distance bounded by T is finite.

We may also view $\mathbb{C}^{N+1}$ as a $2N+2$ dimensional vector space over $\mathbb{R}$. If Λ is a full rank lattice in $\mathbb{R}^{2N+2}$ (that is $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ is isomorphic to $\mathbb{R}^{2N+2}$) then, when T is large, the volume of $T\mathcal{V}_N$ allows for a good approximation to the cardinality of $T\mathcal{V}_N \cap \Lambda$. In this situation knowledge of the volume of $\mathcal{V}_N$ allows for asymptotic estimates for the cardinality of $T\mathcal{V}_N \cap \Lambda$ as $T \to \infty$.

2.2 Multiplicative Distance Functions on Other Algebras

As introduced, multiplicative distance functions provide ways of measuring the complexity of polynomials in $\mathbb{C}[x]$. However our real interest may lie with polynomials whose coefficients are integral, real, or lie in some number field. Of course, since all of these sets of polynomials embed nicely into $\mathbb{C}[x]$ this is a natural setting for multiplicative distance functions.

In spite of this, multiplicative distance functions defined on $\mathbb{R}[x]$ are closer to the spirit of the geometry of numbers since much of the geometry of numbers concerns the relationship between regions of $\mathbb{R}^N$ and lattice points in $\mathbb{Z}^N$. In particular, the calculation of the volume of the star bodies of a multiplicative distance function on $\mathbb{R}[x]$ may have arithmetic significance via techniques from the geometry of numbers. Methods for the computation of volumes of this sort will be presented in Chapter 5.

The notion of multiplicative distance functions may be extended to algebras other than $\mathbb{C}[x]$ and $\mathbb{R}[x]$. For example, consider the set of *conjugate reciprocal* Laurent polynomials. A Laurent polynomial $f(x) \in \mathbb{C}(x)$ is conjugate reciprocal if

$$f\left(\frac{1}{x}\right) = \overline{f(\overline{x})}.$$

Clearly, if $\alpha \in \mathbb{C}$ is a root of f, then $\overline{\alpha^{-1}}$ is also a root. And, if

$$f(x) = c_0 + \sum_{n=0}^{N} c_{-n}x^{-n} + c_n x^n, \quad \text{then} \quad c_{-n} = \overline{c_n},$$

From these facts, it is easy to verify that the set of conjugate reciprocal Laurent polynomials forms an $\mathbb{R}$-algebra. This $\mathbb{R}$-algebra embeds into the set of Laurent polynomials with complex coefficients, though *not* as a subalgebra of $\mathbb{C}[x]$. Nonetheless, any natural distance function on $\mathbb{C}[x]$ will produce a multiplicative distance function on the $\mathbb{R}$-algebra of conjugate reciprocal Laurent polynomials. Generalizations of multiplicative distance functions of this type will not be considered in this thesis.

It should be remarked that there is a simple geometric idea responsible for multiplicative distance functions produced in this section. Let L be any closed linear space embedded in $\mathbb{C}[x]$ which is closed under multiplication, and let Φ be a multiplicative distance function. Then, Φ restricted to L is a multiplicative distance function on the algebra L. If Φ is natural, L can be a closed linear space embedded in the set of Laurent polynomials with complex coefficients which is closed under multiplication. Thus multiplicative distance functions may be created from restricting known multiplicative distance functions to special *slices* of $\mathbb{C}[x]$ or $\mathbb{C}[x, x^{-1}]$. The reciprocal Mahler's measure is formed in this way from the *slice* of reciprocal Laurent polynomials.

2.3 Root Functions

By homogeneity and multiplicativity multiplicative distance functions are completely determined by their action on monic linear polynomials. And thus we introduce the notion of the *root function* associated to a multiplicative distance function. We will show that the class of multiplicative distance functions is completely characterized by the asymptotic behavior of root functions. In so doing we will see that multiplicative distance functions share many similarities with the Mahler's measure.

Let Φ be a multiplicative distance function, and let $\mathbf{a}(x)$ be a polynomial of degree N. There exist N complex numbers $\alpha_1, \ldots, \alpha_N$ so that

$$\mathbf{a}(x) = a_N \prod_{n=1}^{N} (x - \alpha_n).$$

By multiplicativity

$$\Phi(\mathbf{a}) = \Phi(a_N) \prod_{n=1}^{N} \Phi(x - \alpha_n).$$

If we view 1 as a constant polynomial, then clearly $\Phi(1) = \Phi(1)^2$ and thus $\Phi(1) = 1$. Consequently $\Phi(a) = |a|$ and,

$$\Phi(\mathbf{a}) = |a_N| \prod_{n=1}^{N} \Phi(x - \alpha_n).$$

It is clear that $\Phi(x - \alpha)$ depends only on the complex number α, and thus if we define $\phi(\alpha) = \Phi(x - \alpha)$ then

$$\Phi(\mathbf{a}) = |a_N| \prod_{n=1}^{N} \phi(\alpha_n). \tag{2.3}$$

We shall call the function $\phi : \mathbb{C} \to (0, \infty)$ the *root function* associated to Φ. The similarities between Φ and Mahler's measure are immediately obvious from this equation.

If $\phi(\alpha) = \phi(|\alpha|)$ for all $\alpha \in \mathbb{C}$, then we say that Φ is a *radial* multiplicative distance function. Radial multiplicative distance functions measure the complexity of polynomials based on the distances of their roots from the origin. The Mahler's measure is an example of a root radial multiplicative distance function.

As with the Mahler's measure the contribution to Φ by the leading coefficient of a polynomial is of a different nature than the contribution of the roots. The restriction of Φ to the set of monic polynomials plays an important role in the theory of multiplicative distance functions. The *monic restriction* of Φ, given by

$$\widetilde{\Phi}(\mathbf{b}) = \Phi(\mathbf{b}, 1) = \Phi\left(x^N + \sum_{n=0}^{N-1} b_n x^n\right),$$

will play an important role in the study of Φ. If α is a complex number, then $\widetilde{\Phi}(\alpha) = \Phi(x + \alpha)$. Thus the root function for Φ is given explicitly in by $\phi(\alpha) = \widetilde{\Phi}(-\alpha)$.

2.3.1 The Asymptotics of Root Functions

The homogeneity of Φ gives an asymptotic formula for ϕ.

Theorem 2.1. *Suppose* $\Phi : \mathbb{C}[x] \to [0,\infty)$ *is a multiplicative distance function. Then there exists a continuous function* $\phi : \mathbb{C} \to (0,\infty)$ *with* $\phi(\alpha) \sim |\alpha|$ *so that*

$$\Phi : a_N \prod_{n=1}^{N} (x - \alpha_n) \mapsto |a_N| \prod_{n=1}^{N} \phi(\alpha_n).$$

Proof. Since Φ is continuous, non-negative and positive definite, we find that ϕ is continuous and $\phi(\alpha) > 0$ for each $\alpha \in \mathbb{C}$. It remains to show

$$\lim_{|\alpha|\to\infty} \frac{\phi(\alpha)}{|\alpha|} = 1$$

Let a and b be nonzero complex numbers. By homogeneity,

$$\Phi(ax - b) = |a|\, \Phi\left(x - \frac{b}{a}\right) = |a|\, \phi\left(\frac{b}{a}\right)$$

By continuity $\lim_{|a|\to 0} \Phi(ax - b) = \Phi(-b) = |b|$, and thus

$$\lim_{|a|\to 0} |a|\, \phi\left(\frac{b}{a}\right) = |b|.$$

Setting $\alpha = b/a$ we arrive at the statement of the proposition. □

2.3.2 The Continuity of Multiplicative Distance Functions

Suppose we are given a continuous function $\psi : \mathbb{C} \to (0,\infty)$ such that $\psi(\alpha) \sim |\alpha|$. Under what conditions is the function

$$\Psi : a_N \prod_{n=1}^{N} (x - \alpha_n) \mapsto |a_N| \prod_{n=1}^{N} \psi(\alpha_n)$$

a multiplicative distance function? Clearly Ψ is non negative, absolutely homogeneous, positive definite and multiplicative. We need Ψ to be continuous for it to be a distance function. A modification of Mahler's proof that Mahler's measure is continuous reveals that Ψ is continuous. This together with Theorem 2.1 completely categorizes multiplicative distance functions in terms of the asymptotic behavior of their root functions.

Theorem 2.2. *Suppose $\psi : \mathbb{C} \to (0, \infty)$ is a continuous function such that $\psi(\alpha) \sim |\alpha|$ then the function $\Psi : \mathbb{C}[x] \to [0, \infty)$ given by*

$$\Psi : a_N \prod_{n=1}^{N} (x - \alpha_n) \mapsto |a_N| \prod_{n=1}^{N} \psi(\alpha_n)$$

is a multiplicative distance function. In fact, in this situation Ψ is continuous with respect to the strong topology on $\mathbb{C}[x]$.

Corollary 2.3. *Every multiplicative distance function is continuous with respect to the strong topology topology on $\mathbb{C}[x]$.*

We mention the special case of the continuity of Mahler's measure since it answers a question asked by Mahler in [10] (see the remark following Lemma 1).

Corollary 2.4. *Mahler's measure is continuous with respect to the strong topology on $\mathbb{C}[x]$.*

Theorem 2.2 is dependent on Hurwitz's Root Theorem. Hurwitz's Root Theorem is a consequence of Rouché's Theorem; A proof of Hurwitz's Root Theorem can be found in [14].

Theorem 2.5 (Hurwitz's Root Theorem)**.** *Let K be a closed set of $\mathbb{C}$, and suppose $\{f_k(z)\}$ is a sequence of functions which are continuous on K and holomorphic in the interior of K. Suppose further that $\{f_k(z)\}$ is uniformly convergent on K and that $f(z) = \lim_{k\to\infty} f_k(z)$ vanishes nowhere on the boundary of K. Then there exists a positive integer k_0 such that if $k > k_0$ then $f_k(z)$*

and $f(z)$ have the same number of zeros (counted according to multiplicity) in the interior of K.

Corollary 2.6. *Suppose $\{f_k(x)\}$ is a sequence of polynomials in $\mathbb{C}[x]$ such that*

$$f_k(x) = a_{kN_k} \prod_{n=1}^{N_k} (x - \alpha_{kn}) \quad \text{for} \quad k = 1, 2, \ldots,$$

and suppose $\{f_k(x)\}$ converges to

$$f(x) = a_N \prod_{n=1}^{N} (x - \alpha_n),$$

in the strong topology on $\mathbb{C}[x]$. Then, for each $\epsilon > 0$ there exists a positive integer k_0 such that for each $k > k_0$ there is a reordering of $\{\alpha_{k1}, \ldots, \alpha_{kN_k}\}$ with

$$|\alpha_{kn} - \alpha_n| < \epsilon \quad \text{for} \quad n = 1, \ldots, N,$$

and

$$|\alpha_{kn}| \geq \frac{1}{\epsilon} \quad \text{for} \quad n = N+1, \ldots, N_k.$$

Proof. It shall be convenient to relabel the roots of $f(x)$ so that $\beta_1, \ldots, \beta_M$ are the distinct roots of $f(x)$ with multiplicities $\eta_1, \ldots, \eta_M$. Let

$$\epsilon_1 = \min \left\{ \frac{|\beta_j - \beta_m|}{2} : j, m = 1, \ldots, M \quad j \neq m \right\}, \quad \text{if} \quad M > 1,$$

and set $\epsilon_1 = \epsilon$ if $M = 1$.

Define $K_1, \ldots, K_M$ to be the closed subsets of $\mathbb{C}$ given by

$$K_m = \big\{ z : |z - \beta_m| \leq \min\{\epsilon, \epsilon_1\} \big\} \quad \text{for} \quad m = 1, \ldots, M.$$

Notice that the interiors of $K_1, \ldots, K_M$ are pairwise disjoint. Next, let K_∞ be the closed subset of $\mathbb{C}$ given by

$$K_\infty = \{ z : |z| \leq 1/\epsilon \} \cup \bigcup_{m=1}^{M} K_m.$$

For each $m = 1, \ldots, M$, Hurwitz's Root Theorem guarantees the existence of an integer k_m such that if $k > k_m$ then $f_k(x)$ has exactly η_m roots in the interior of K_m. Similarly, there exists an integer k_∞ such that if $k > k_\infty$ then $f_k(x)$ has exactly N roots in the interior of K_∞. It follows that the remaining $N_k - N$ roots of $f_k(x)$ have modulus $\geq 1/\epsilon$.

The corollary follows by setting $k_0 = \max\{k_1, \ldots, k_N, k_\infty\}$, and for $k > k_0$, reordering the roots of $f_k(x)$ according to their inclusions in $K_1, \ldots, K_M$ and the complement of K_∞. □

We are now in position to prove Theorem 2.2.

Proof of Theorem 2.2. It is clear from construction and the properties of ψ that Ψ is non-negative, absolutely homogeneous and positive definite. It remains to show that Ψ is continuous. Suppose $\{f_k(x)\}$ is a sequence of polynomials in $\mathbb{C}[x]$ such that

$$f_k(x) = a_{kN_k} \prod_{n=1}^{N_k} (x - \alpha_{kn}) \quad \text{for} \quad k = 1, 2, \ldots,$$

and

$$\lim_{k \to \infty} f_k(x) = f(x) = a_N \prod_{n=1}^{N} (x - \alpha_n),$$

uniformly on compact subsets of $\mathbb{C}$. We will show that

$$\lim_{k \to \infty} \Psi(f_k) = \Psi(f).$$

By the corollary to Hurwitz's Root Theorem we may reorder the roots of each $f_k(x)$ so that

$$\lim_{k \to \infty} \alpha_{kn} = \alpha_n \quad \text{for} \quad n = 1, \ldots, N.$$

For each $k > 0$, define the polynomials $g_k(x)$ and $h_k(x)$ by,

$$g_k(x) = a_N \prod_{n=1}^{N}(x - \alpha_{kn}) \quad \text{and} \quad h_k(x) = \frac{a_{kN_k}}{a_N} \prod_{n=N+1}^{N_k} (x - \alpha_{kn}),$$

Notice that $g_k(x) \to f(x)$ as $k \to \infty$. Furthermore, since $g_k(x)$ is of degree N for all k, it follows that this convergence is uniform on compact subsets of $\mathbb{C}$. Now,

$$\begin{aligned} |g_k(x)|\,|h_k(x) - 1| &= |f_k(x) - g_k(x)| \\ &\leq |f_k(x) - f(x)| + |f(x) - g_k(x)|\,, \end{aligned}$$

from which it follows that $\{h_k(x)\}$ converges to the constant function 1 uniformly on compact subsets of $\mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_N\}$.

The constant coefficient of $h_k(x)$ is given by

$$\frac{a_{N_k}}{a_N}(-1)^{N_k - N} \prod_{n=N+1}^{N_k} \alpha_{kn},$$

and thus, by choosing a point $x \in \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_N\}$ and using the fact that $h_k(x) \to 1$ we have,

$$\lim_{k\to\infty} \left\{ \frac{|a_{N_k}|}{|a_N|} \prod_{n=N+1}^{N_k} |\alpha_{kn}| \right\} = 1. \tag{2.4}$$

This is the key fact needed to prove the theorem:

$$\begin{aligned} \lim_{k\to\infty} \Psi(f_k) &= \lim_{k\to\infty} \left(|a_{kN_k}| \prod_{n=1}^{N_k} \psi(\alpha_{kn}) \right) \\ &= \lim_{k\to\infty} \left(\left\{ \frac{|a_{kN_k}|}{|a_N|} \prod_{n=N+1}^{N_k} \psi(\alpha_{kn}) \right\} \left\{ |a_N| \prod_{n=1}^{N} \psi(\alpha_{kn}) \right\} \right) \\ &= \lim_{k\to\infty} \left(\left\{ \frac{|a_{kN_k}|}{|a_N|} \prod_{n=N+1}^{N_k} |\alpha_{kn}| \right\} \left\{ |a_N| \prod_{n=1}^{N} \psi(\alpha_{kn}) \right\} \right). \end{aligned}$$

Where the last equation is a consequence of the fact that $|\alpha_{kn}| \to \infty$ as $k \to \infty$, and $\psi(\alpha) \sim |\alpha|$. From Equation 2.4 it follows that,

$$\lim_{k\to\infty} \Psi(f_k) = \lim_{k\to\infty} \left\{ |a_N| \prod_{n=1}^{N} \psi(\alpha_{kn}) \right\} = |a_N| \prod_{n=1}^{N} \psi(\alpha_n) = \Psi(f),$$

where the second equality follows from the continuity of ψ and the fact that $\lim_{k\to\infty} \alpha_{kn} = \alpha_n$. $\square$

2.4 Examples of Multiplicative Distance Functions

The only concrete example of a multiplicative distance function we have seen so far is Mahler's measure. The Mahler's measure was the first (and arguably the only) multiplicative distance function to receive much attention. This attention is not undeserved since in a certain sense the Mahler's measure is the simplest natural multiplicative distance function: If Φ is a natural multiplicative distance function with root function ϕ, then ϕ is continuous, $\phi(\alpha) \sim |\alpha|$, and $\phi(0) = 1$. The root function $\phi(\alpha) = \max\{1, |\alpha|\}$ is arguably the simplest function that satisfies these criteria.

In this section, we will introduce more examples of multiplicative distance functions using Theorem 2.2. In particular, we will formally introduce the reciprocal Mahler's measure. All of the multiplicative distance functions in this section will be specified by their root functions. We will eventually see examples of multiplicative distance functions that are determined by their action on coefficient vectors. However, since these multiplicative distance functions arise from a seemingly different mechanism we will reserve these for the next chapter.

Example 2.4.1 (the q-reciprocal Mahler's measure)**.** Let $q \in [0, \infty)$ and define

$\phi_q : \mathbb{C} \to (0, \infty)$ by

$$\phi_q(\alpha) = \max\left\{1, \left|\frac{\alpha + \sqrt{\alpha^2 - 4q}}{2}\right|\right\} \max\left\{1, \left|\frac{\alpha - \sqrt{\alpha^2 - 4q}}{2}\right|\right\}.$$

where $\sqrt{\cdot}$ denotes any fixed branch of the square root function. ϕ_q is independent of the branch of the square root chosen since both $\sqrt{\cdot}$ and $-\sqrt{\cdot}$ appear in the definition. Define the multiplicative distance function μ_q by

$$\mu_q : a_N \prod_{n=1}^{N} (x - \alpha_n) \mapsto |a_N| \prod_{n=1}^{N} \phi_q(\alpha_n). \tag{2.5}$$

This is indeed a multiplicative distance function since ϕ_q is positive, continuous and $\phi_q \sim |\alpha|$. Furthermore, when $q \leq 1$, μ_q is a natural distance function. Notice that the Mahler's measure is equal to μ_0.

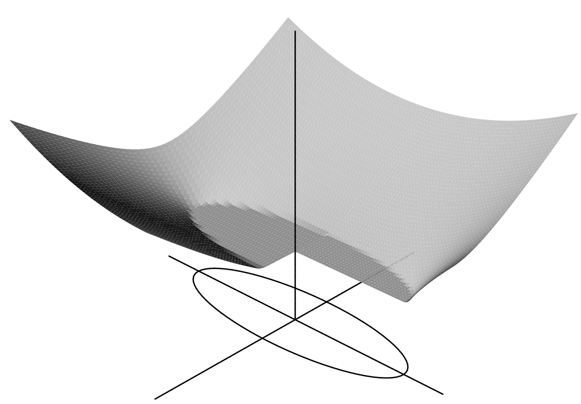

Figure 2.1: A plot of ϕ_q for $q = .5$ near the origin

Figure 2.1 shows a plot of $\phi_{.5}$ near the origin. From the figure we can see an elliptical region where ϕ_q takes the value 1. This region is indeed bounded by an ellipse. The appearance of this ellipse is not accidental (nor restricted to the $q = .5$ case). In the next chapter we will see how to create multiplicative distance functions associated to certain compact subsets of $\mathbb{C}$, and we will see that the μ_q are formed in this manner for a family of ellipses indexed by q.

When $q = 1$ we arrive at a very important distance function, the *reciprocal Mahler's measure*. Explicitly,

$$\mu_1 : a_N \prod_{n=1}^{N} (x - \alpha_n) \mapsto |a_N| \prod_{n=1}^{N} \max\left\{ \left| \frac{\alpha_n \pm \sqrt{\alpha_n^2 - 4}}{2} \right| \right\}. \tag{2.6}$$

As remarked previously, μ_1 is related to the Mahler's measure restricted to reciprocal Laurent polynomials. In fact, when $q \leq 1$, μ_q is related to Mahler's measure restricted to a set of Laurent polynomials determined by q. To see this let $f(x) = a_N \prod_{n=1}^{N}(x - \alpha_n)$, then

$$\begin{aligned} f\left(x + \frac{q}{x}\right) &= a_N \prod_{n=1}^{N} \left(x + \frac{q}{x} - \alpha_n\right) \\ &= a_N x^{-N} \prod_{n=1}^{N} (x^2 - \alpha_n x + q) \\ &= a_N x^{-N} \prod_{n=1}^{N} \left(x - \frac{\alpha_n + \sqrt{\alpha_n^2 - 4q}}{2}\right)\left(x - \frac{\alpha_n - \sqrt{\alpha_n^2 - 4q}}{2}\right). \end{aligned}$$

It follows from the definition of μ_q and Equation 2.2, that

$$\mu_q(f(x)) = \mu\left(f\left(x + \frac{q}{x}\right)\right).$$

Thus μ_q can be regarded as the Mahler's measure restricted the set of Laurent polynomials given by $\mathbb{C}[x + q/x]$. In particular, μ_1 can be regarded as Mahler's measure restricted to reciprocal Laurent polynomials.

Example 2.4.2 (smooth approximations to Mahler's measure)**.** We introduce another family of multiplicative distance functions related to the Mahler's measure. This family of distance functions has the nice feature that the root functions are radial and differentiable.

Let m be a positive parameter, let $\omega_m : \mathbb{C} \to (0, \infty)$ be given by

$$\omega_m(\alpha) = (1 + |\alpha|^m)^{1/m},$$

and let Ω_m be the multiplicative distance function given by

$$\Omega_m : a_N \prod_{n=1}^{N} (x - \alpha_n) \mapsto |a_N| \prod_{n=1}^{N} \omega_m(\alpha_n).$$

Ω_m is natural and root radial. Furthermore

$$\lim_{m \to \infty} \omega_m(\alpha) = \max\{1, |\alpha|\},$$

and thus, for each f in $\mathbb{C}[x]$

$$\mu(f) = \lim_{m \to \infty} \Omega_m(f).$$

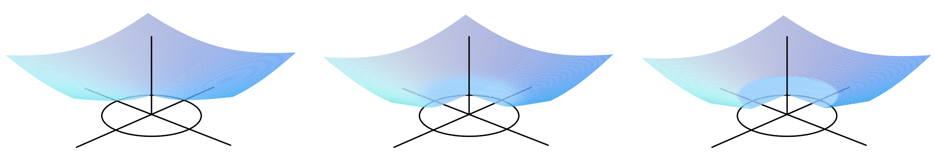

Figure 2.2: plots of ω_2, ω_{10} and $\max\{1, |\alpha|\}$ near the unit circle

It is easy to make other multiplicative distance functions by appealing to Theorem 2.2. The examples produces here are of arithmetic interest since they are associated to Mahler's measure.

Chapter 3

Potentials and Jensen's Formula

For any $\alpha \in \mathbb{C}$ and $r > 0$,

$$\int_0^1 \log \left| re^{2\pi i\theta} - \alpha \right| \, d\theta = \log \max\{r, |\alpha|\}.$$

This formula is known as Jensen's formula. See [6] for a proof (or three). As an immediate consequence, if $f(x) = a_N \prod_{n=1}^N (x - \alpha_n)$, then

$$\exp\left\{ \int_0^1 \log \left| f\left(e^{2\pi i\theta}\right) \right| \, d\theta \right\} = |a_N| \prod_{n=1}^N \max\{1, |\alpha_n|\}. \tag{3.1}$$

The appearance of both coefficients and roots in this equation underscores the importance of Mahler measure as a measure of complexity of polynomials. In Chapter 2 we generalized the right hand side of this equation in order to produce new multiplicative distance functions. In this chapter we shall generalize the left hand side to produce new multiplicative distance functions. In fact, we will see that both the q-reciprocal Mahler measure, and the smooth approximations to Mahler measure satisfy equations analogous to Equation 3.1. We will say that the multiplicative distance function Φ possesses a Jensen's formula if there exists a probability measure ν on $\mathbb{C}$ such that for each $f \in \mathbb{C}[x]$,

$$\Phi(f) = \exp\left\{ \int_{\mathbb{C}} \log |f(w)| \, d\nu(w) \right\}.$$

We will see that Jensen's original formula is a special case of a more general phenomenon, and it begs the question: Which multiplicative distance functions have Jensen's formulae?

As a partial answer to this question we will present two classes of multiplicative distance functions which possess Jensen's formulae. One of these classes is the set of multiplicative distance functions whose root functions are radial and smooth. We will also produce a class of multiplicative distance functions via potential theory which have Jensen's formulae. The potential theoretic approach is very powerful, and will allow us to associate multiplicative distance functions to compact subsets of $\mathbb{C}$. By concentrating on compact subsets of $\mathbb{C}$ of arithmetic interest, our method will produce multiplicative distance functions which can answer questions of interest to number theorists.

As usual, Mahler measure plays a central role in our discussion of multiplicative distance functions; Mahler measure can be constructed from a special potential associated to the closed unit disk. The reciprocal Mahler measure too is associated to a compact subset of $\mathbb{C}$, the interval $[-2, 2]$ on the real axis. As the parameter q varies from 0 to 1, we shall see that the q-reciprocal Mahler measure can be associated to a family of ellipses which interpolate between the closed unit disk and the interval $[-2, 2]$ on the real axis. As q varies, the right hand side of Equation 3.1 changes to reflect the fact that we are studying Mahler measure restricted to $\mathbb{C}[x + q/x]$. This is no accident, and we shall see that Mahler measure restricted to certain subalgebras of $\mathbb{C}[x]$ can be expressed as multiplicative distance functions determined from other special compact subsets of $\mathbb{C}$.

3.1 Jensen's Formulae for Smooth Radial Root Functions

Before diving headlong into potential theory we will first prove results about multiplicative distance functions with smooth radial root functions. This will give us a generalization of Jensen's formula for our family of smooth approximations to Mahler measure.

Theorem 3.1. *Suppose Φ is a multiplicative distance function with radial root function ϕ. Suppose further that $\phi(r)$ is twice differentiable on $(0,\infty)$. Then, there exists a function $u : (0,\infty) \to \mathbb{R}$ so that*

$$\Phi(f) = \exp\left\{\int_{\mathbb{C}} \log|f(w)|\, u(|w|) d\lambda_2(w)\right\} \quad \text{for} \quad f(x) \in \mathbb{C}[x].$$

Moreover,

$$u(t) = \frac{1}{2\pi t}\frac{d}{dt}\left\{t\frac{\phi'(t)}{\phi(t)}\right\} = \frac{1}{2\pi}\left(\frac{\phi(t)\phi''(t) - \phi'(t)^2}{\phi(t)^2} + \frac{\phi'(t)}{t\phi(t)}\right).$$

Proof. By multiplicativity it suffices to find a function $u : (0,\infty) \to \mathbb{R}$ so that

$$\log\phi(\alpha) = \int_{\mathbb{C}} \log|w-\alpha|\, u(|w|)\, d\lambda_2(w)$$

Making the substitution $\alpha = re^{2\pi i\theta}$ we find

$$\begin{aligned}
\log\phi(\alpha) &= 2\pi\int_0^\infty\int_0^1 \log|re^{2\pi i\theta} - \alpha|\, ru(r)\, d\theta\, dr \\
&= 2\pi\int_0^\infty ru(r)\left\{\int_0^1 \log|re^{2\pi i\theta} - \alpha|\, d\theta\right\} dr \\
&= 2\pi\int_0^\infty ru(r)\log\max\{r, |\alpha|\}\, dr.
\end{aligned}$$

Where the last equation follows from Jensen's formula. Letting $t = |\alpha|$ and since ϕ is radial we may write

$$\log\phi(t) = 2\pi\log t\int_0^t ru(r)\, dr + 2\pi\int_t^\infty ru(r)\log r\, dr.$$

By differentiating both sides of this equation we find

$$\frac{\phi'(t)}{\phi(t)} = \frac{2\pi}{t}\int_0^t ru(r)\, dr.$$

Differentiating again we find

$$\frac{d}{dt}\left\{t\frac{\phi'(t)}{\phi(t)}\right\} = 2\pi\, tu(t),$$

and the theorem follows. $\square$

Example 3.1.1. Recall that for each $m > 0$ we define Ω_m to be the multiplicative distance function whose root function is given by $\omega_m(\alpha) = (1+|\alpha|^m)^{1/m}$. It follows that Ω_m satisfies the conditions of Theorem 3.1, and a short calculation reveals that

$$\log \omega_m(\alpha) = \frac{m}{2\pi} \int_{\mathbb{C}} \log |w - \alpha| \frac{|w|^{m-2}}{(1+|w|^m)^2} \, d\lambda_2(w),$$

and hence,

$$\Omega_m(f) = \exp \left\{ \frac{m}{2\pi} \int_{\mathbb{C}} \log |f(w)| \frac{|w|^{m-2}}{(1+|w|^m)^2} \, d\lambda_2(w) \right\}.$$

Let $u_m : [0, \infty) \to \mathbb{R}$ be the function given by

$$u_m(t) = \frac{m\, t^{m-2}}{2\pi(1+t^m)^2},$$

and let ν_m be the measure on $\mathbb{C}$ given by $d\nu_m(w) = u_m(|w|)\, d\lambda_2(w)$. It is easy to verify that ν_m is a probability measure on $\mathbb{C}$. And, since $\lim_{m\to\infty} \Omega_m = \mu$ pointwise on $\mathbb{C}[x]$, it is reasonable to expect that as $m \to \infty$ the measures ν_m approach (in some sense) Lebesgue measure on the unit circle. It is *not* reasonable to expect ν_m to converge weakly to Lebesgue measure on the unit circle since the support of ν_m is all of $\mathbb{C}$. Nonetheless, we can explore the qualitative nature of the measures ν_m as $m \to \infty$. Figure 3.2 shows a density plot of ν_m for some small values of m. Notice how as m increases the mass of ν_m accumulates around the unit circle.

3.2 Potential Theory

In the last section we used the root functions of certain multiplicative distance functions to create a measure on $\mathbb{C}$ which was then used to produce a Jensen's formula. In this section we will employ the opposite approach: We will begin with a measure and demonstrate that under certain hypotheses it can be used

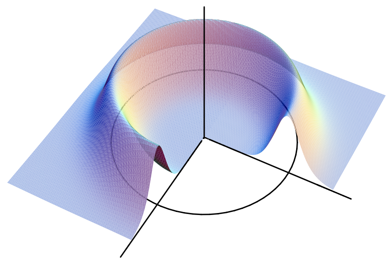

Figure 3.1: A plot of u_{10} near the unit circle

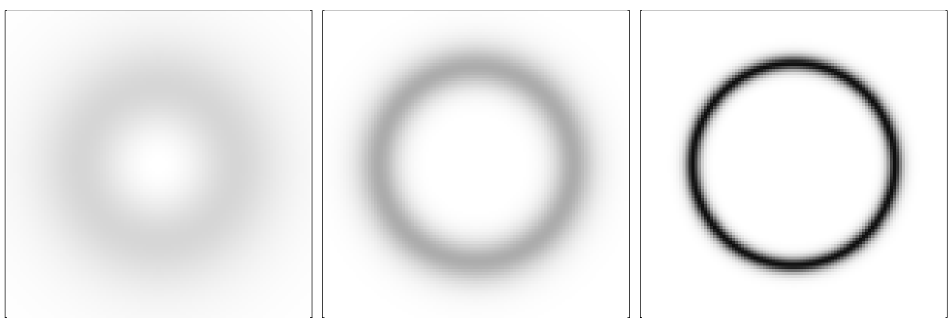

Figure 3.2: Density plots of ν_4, ν_{10} and ν_{30} near the unit circle

to construct a multiplicative distance function. We will employ potential theory in order to create new (and old!) multiplicative distance functions. We will also discover how to find (at least theoretically) the Jensen's formula of a potential-theoretic multiplicative distance function.

Many standard results of potential theory are contained in this chapter. Our goal is to employ potentials to study multiplicative distance functions, and the presentation here will reflect this purpose. As such, many of the results of potential theory are stated and proved under the specialized (and weaker) hypotheses pertinent in the study of multiplicative distance functions. The reader who is interested in the intricacies of potential theory is referred to

Ransford's excellent introductory text [12].

3.2.1 Subharmonic and Harmonic Functions

Before introducing potentials we first introduce subharmonic functions. Let U be an open subset of $\mathbb{C}$, and $u : U \to [-\infty, \infty)$ be a function which is not identically $-\infty$. We shall say u is *upper semicontinuous* at $\alpha \in U$ if

$$\limsup_{z \to \alpha} u(z) \leq u(\alpha).$$

If u is upper semicontinuous at every $\alpha \in U$ we call it upper semicontinuous on U. It is an easy exercise to show that if u is upper semicontinuous on U and K is a compact subset of U then u is bounded above and attains its maximum on K.

If there exists a $\rho = \rho(\alpha) > 0$ (dependent on α), such that

$$u(\alpha) \leq \frac{1}{2\pi} \int_0^{2\pi} u(\alpha + re^{i\theta})\, d\theta \quad (0 \leq r < \rho), \tag{3.2}$$

then we shall say that u satisfies the *submean inequality* at α. The submean inequality guarantees that $f(\alpha)$ is smaller than the average value taken by f on the boundary of a disk of sufficiently small radius surrounding α. The submean inequality is analogous to that satisfied by locally convex functions: If $f : (a, b) \to [-\infty, \infty)$ is locally convex at $x \in (a, b)$, then there exists an $r > 0$ such that

$$f(x) \leq \frac{f(x - \epsilon) + f(x + \epsilon)}{2} \quad 0 \leq \epsilon < r,$$

and $f(x)$ is smaller than the average value of f taken on an interval of sufficiently small length surrounding x.

If u is upper semicontinuous and satisfies the submean inequality at every α in U then we shall say that u is *subharmonic* on U. If both u and $-u$

are subharmonic on U, then we will say that u is *harmonic* on U. Notice that harmonic functions are continuous and for every $\alpha \in U$ there is equality in the submean inequality.

We give some examples of harmonic and subharmonic functions which are important for our cause. For the next three examples assume $g : U \to \mathbb{C}$ is a holomorphic on U.

Example 3.2.1. $\Re(g)$ and $\Im(g)$ are harmonic on U. To see this let $\Delta = \Delta(\alpha; r)$ be a disk surrounding α which is completely contained in U. Clearly $\Re(g)$ is continuous, and Cauchy's formula shows that

$$g(\alpha) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{g(z)}{z-\alpha}\,dz = \frac{1}{2\pi}\int_0^{2\pi} g(\alpha + re^{i\theta})\,d\theta.$$

Taking the real part of both sides of this equation shows that $\Re(g)$ is harmonic. A similar argument shows that $\Im(g)$ is also harmonic on U.

Example 3.2.2. The function $\alpha \mapsto |g(\alpha)|$ is subharmonic on U. Indeed, continuity is obvious, and if r is sufficiently small then

$$|g(\alpha)| = \frac{1}{2\pi}\left|\int_0^{2\pi} g(\alpha + re^{i\theta})\,d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} \left|g(\alpha + re^{i\theta})\right|\,d\theta$$

Example 3.2.3. The function is $\alpha \mapsto \log|g(\alpha)|$ is subharmonic on $\mathbb{C}$. To see this notice that if $g(\alpha) \neq 0$ then $\log g$ is holomorphic in a neighborhood of α. If $V \subseteq U$ is a simply connected open set containing no zeros of g, then there exists a holomorphic function $h : V \to \mathbb{C}$ such that $g = e^h$. It follows that $\log|g| = \Re(h)$, and thus $\log|g|$ is harmonic except at the zeros of g. If α is a zero of g, then Equation 3.2 is trivially satisfied. The upper semicontinuity of $\log|g|$ is clear since $\lim_{z\to\alpha} \log|g(z)| = \log|g(\alpha)|$ for all $\alpha \in U$ (even when $\log|g(\alpha)| = -\infty$).

Example 3.2.4. If u and v are both subharmonic on the domain U, then it is easily verified that $\alpha \mapsto \max\{u(\alpha), v(\alpha)\}$ is also subharmonic on U. In particular, $\alpha \mapsto \max\{1, |\alpha|\}$ is a subharmonic function.

One powerful feature of subharmonic functions is that they satisfy analogs of Liouville's theorem and the maximum modulus principle for holomorphic functions.

Theorem 3.2 (Maximum Principle)**.** *Let D be a domain of $\mathbb{C}$, and let $u : D \to [-\infty, \infty)$ be subharmonic. Then,*

1. *If u attains a global maximum on D, then u is constant.*
2. *If $\limsup_{z\to\zeta} u(z) \leq 0$ for all $\zeta \in \partial D$, then $u \leq 0$ on D.*

Remark 3.2.1. If D is unbounded, then it is conventional to consider $\infty \in \partial D$.

Proof. For part 1, suppose u attains its global maximum M on U, and define

$$A = \{z \in D : u(z) < M\} \quad \text{and} \quad B = \{z \in D : u(z) = D\}.$$

By the upper semicontinuity of u, A is open. Also, B is open, since if $u(w) = M$, then the submean inequality at w forces u to be equal to M on all sufficiently small circles around w. Clearly $D = A \cup B$, and since U is connected, either $D = A$ or $D = B$. By assumption B is nonempty, and hence $D = B$.

To prove part 2, extend u to ∂D by specifying

$$u(\zeta) = \limsup_{z\to\zeta} u(z) \quad \text{for} \quad \zeta \in \partial D,$$

Then, u is upper semicontinuous on the closure of D, $\overline{D}$. Clearly, $\overline{D}$ is a compact subset of the extended complex numbers, and it follows that u attains its maximum at some point $w \in \overline{D}$. If $w \in \partial D$, then by assumption $u(w) \leq 0$, and consequently $u \leq 0$ on D. If $w \in D$, then by part 1, u is constant on D, hence on ∂D, and thus by assumption $u \leq 0$ on D. $\square$

3.2.2 Potentials and Logarithmic Potentials

An important class of subharmonic functions are potentials. Let ν be a finite Borel measure on $\mathbb{C}$ which is supported on the compact set K. The *potential* of ν is defined to be the function $p_\nu : \mathbb{C} \to [0, \infty)$ given by

$$p_\nu(\alpha) = \exp\left\{\int_{\mathbb{C}} \log|w - \alpha| \, d\nu(w)\right\}.$$

This quantity is called the potential because it arises in the study of potential energy of certain physical systems. It shall sometimes be convenient to work with the *logarithmic potential* of ν which is given by $\log p_\nu$. A word of caution: Many authors use the notation p_ν for the logarithmic potential of ν. The notation used here is more convenient for our purposes since, as we shall see, potentials often form root multiplicative distance functions.

As a first step we will show that logarithmic potentials are subharmonic on $\mathbb{C}$.

Theorem 3.3. *Let ν be a finite Borel measure on $\mathbb{C}$ with compact support. Then, $\log p_\nu$ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus \operatorname{supp}\nu$.*

Proof. Since subharmonicity is a local condition, it suffices to demonstrate that $\log p_\nu$ is subharmonic on every relatively compact open subset U of $\mathbb{C}$.

Define the function $v : \mathbb{C} \times \mathbb{C} \to [-\infty, \infty)$ by $v(\alpha, w) = \log|w - \alpha|$. By our previous remarks on subharmonic functions, $v(\alpha, w)$ is subharmonic and hence upper semicontinuous in each variable. It follows that there exists a constant c such that $v(\alpha, w) < c$ on $\overline{U} \times \operatorname{supp}\nu$. Now, since $v(\alpha, w) - c$ is negative on $\overline{U} \times \operatorname{supp}\nu$, by Fatou's lemma we find that

$$\begin{aligned}
\limsup_{z\to\alpha} \log p_\nu(z) - c &= \limsup_{z\to\alpha} \int_{\operatorname{supp}\nu} v(z, w) - c \, d\nu(w) \\
&\le \int_{\operatorname{supp}\nu} \limsup_{z\to\alpha} v(z, w) - c \, d\nu(w) = \log p_\nu(\alpha) - c.
\end{aligned}$$

Thus, $\log p_\nu$ is upper semicontinuous.

Now for each $\alpha \in U$ there exists a $\rho > 0$ such that

$$\log|w-\alpha| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|w-\alpha+re^{i\theta}|\,d\theta \quad (0 \leq r < \rho),$$

with equality when $w \neq \alpha$. It follows that, when $0 \leq r < \rho$,

$$\begin{aligned}\log p_\nu(\alpha) = \int_{\mathbb{C}} \log|w-\alpha|\,d\nu(w) &\leq \frac{1}{2\pi}\int_{\mathbb{C}}\int_0^{2\pi} \log|w-\alpha+re^{i\theta}|\,d\theta\,d\nu(w)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}} \log|w-\alpha+re^{i\theta}|\,d\nu(w)\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} \log p_\nu(\alpha+re^{i\theta})d\theta.\end{aligned}$$

And thus, $\log p_\nu$ satisfies the submean inequality. Moreover, there is equality in this expression if $\alpha \notin \operatorname{supp}\nu$. We conclude that $\log p_\nu$ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus \operatorname{supp}\nu$. □

We can now present one the principle results of this chapter: Under mild conditions, potentials are root functions of multiplicative distance functions. This provides a method of creating new multiplicative distance functions by generalizing the left hand side of Equation 3.1.

Theorem 3.4. *Let ν be a Borel probability measure on $\mathbb{C}$ with compact support such that $\log p_\nu$ is continuous. Then the function $P_\nu : \mathbb{C}[x] \to [0,\infty)$ given by*

$$P_\nu(f) = \exp\left\{\int_{\mathbb{C}} \log|f(w)|\,d\nu(w)\right\}$$

is a multiplicative distance function.

The proof of this theorem relies on a lemma giving an asymptotic description of potentials.

Lemma 3.5. *Let ν be a Borel probability measure on $\mathbb{C}$ with compact support. Then, $p_\nu(\alpha) \sim |\alpha|$ as $|\alpha| \to \infty$.*

Proof. Let K be the support of ν. Then,

$$\begin{aligned} p_\nu(\alpha) &= \exp\left\{\int_K \log|w-\alpha|\,d\nu(w)\right\} \\ &= \exp\left\{\log|\alpha| + \int_K \log|1-w/\alpha|\,d\nu(w)\right\}. \end{aligned}$$

It follows that

$$\lim_{|\alpha|\to\infty} \frac{p_\nu(\alpha)}{|\alpha|} = \lim_{|\alpha|\to\infty} \exp\left\{\int_K \log|1-w/\alpha|\,d\nu(w)\right\}.$$

Now let $c = \sup\limits_{w\in K} |w|$. It follows that for all $w \in K$

$$\log|1-w/\alpha| \le \log(1+c/|\alpha|).$$

Thus, if $|\alpha| > c$ then $\log|1-w/\alpha| \le \log 2$, and by the dominated convergence theorem,

$$\lim_{|\alpha|\to\infty} \exp\left\{\int_K \log|1-w/\alpha|\,d\nu(w)\right\} = 1.$$

And consequently, $p_\nu(\alpha) \sim |\alpha|$ as $|\alpha| \to \infty$. □

Lemma 3.5 demonstrates that potentials satisfy the same asymptotic formula as root functions of multiplicative distance functions.

Proof of Theorem 3.4. If $f(x) = a_N \prod\limits_{n=1}^{N} (x-\alpha_n)$, then clearly

$$P_\nu(f) = |a_N| \prod_{n=1}^{N} p_\nu(\alpha).$$

Since $\log p_\nu$ is continuous it follows that p_ν is continuous and positive on all of $\mathbb{C}$. Thus, by Lemma 3.5 and Theorem 2.2, P_ν is a multiplicative distance function. □

So far, nothing has been said about the continuity of logarithmic potentials except that it is critical for the purposes of creating multiplicative distance functions. In general, we cannot say much about the continuity of logarithmic potentials since there do exist logarithmic potentials which are upper semicontinuous but not continuous. However, the next theorem does provide a useful condition for the continuity of logarithmic potentials.

Theorem 3.6 (Continuity Principle)**.** *Let ν be a finite Borel measure on $\mathbb{C}$ with compact support K. Suppose for $\zeta_0 \in K$*

$$\lim_{\substack{\zeta\to\zeta_0\\ \zeta\in K}} \log p_\nu(\zeta) = \log p_\nu(\zeta_0).$$

Then,

$$\lim_{z\to\zeta_0} \log p_\nu(z) = \log p_\nu(\zeta_0).$$

Proof. Clearly the theorem is trivial unless $\zeta_0 \in \partial K$.

It suffices to prove that

$$\liminf_{z\to\zeta_0} \log p_\nu(z) = \liminf_{\substack{\zeta\to\zeta_0\\ \zeta\in K}} \log p_\nu(\zeta),$$

since, if this holds then

$$\liminf_{z\to\zeta_0} \log p_\nu(z) = \log p_\nu(\zeta_0),$$

and by upper semicontinuity of $\log p_\nu$,

$$\limsup_{z\to\zeta_0} \log p_\nu(z) \le \log p_\nu(\zeta_0),$$

If $\log p_\nu(\zeta_0) = -\infty$, then $\lim_{z\to\zeta_0} p_\nu(z) = \log p_\nu(\zeta_0)$ by the upper semicontinuity of $\log p_\nu$. We may thus assume that $\log p_\nu(\zeta_0) \ne -\infty$, and thus that $\nu(\{\zeta_0\}) = 0$. Thus, given $\epsilon > 0$, We may find an open neighborhood U of ζ_0 such that $\nu(U) \le \epsilon$.

Now, given $z \in \mathbb{C}$, choose $\zeta \in K$ such that $|\zeta - z|$ is minimized. Then, for all $w \in K$,

$$\frac{|\zeta - w|}{|z - w|} \le \frac{|\zeta - z| + |z - w|}{|z - w|} \le 2.$$

It follows that

$$\begin{aligned}\log p_\nu(z) &= \log p_\nu(\zeta) - \int_K \log \frac{|\zeta - w|}{|z-w|}\, d\nu(w) \\ &\geq \log p_\nu(\zeta) - \int_{K\setminus U} \log \frac{|\zeta - w|}{|z-w|}\, d\nu(w) - \epsilon \log 2\end{aligned}$$

As $z \to \zeta_0 \in \mathbb{C}$ then also $\zeta \to \zeta_0 \in K$, and hence

$$\log \frac{|\zeta - w|}{|z-w|} \to 0 \quad \text{as} \quad z \to \zeta_0.$$

And thus

$$\liminf_{z\to\zeta_0} \log p_\nu(z) = \liminf_{\substack{\zeta\to\zeta_0 \\ \zeta\in K}} \log p_\nu(\zeta) - \epsilon \log 2.$$

And the result follows since ϵ is arbitrary. $\square$

Remark 3.2.2. Generally, potentials are not continuous but upper semicontinuous. And, while weaker than continuity, upper semicontinuity is still a powerful condition. As such, it may be profitable to relax the axioms of multiplicative distance functions to include those whose root functions are upper semicontinuous.

The continuity principle is key to another important result about logarithmic potentials.

Theorem 3.7 (Minimum Principle)**.** *Let ν be a finite Borel measure on $\mathbb{C}$ with compact support K. If there is a constant M such that $\log p_\nu \geq M$ on K, then $\log p_\nu \geq M$ on all of $\mathbb{C}$.*

Proof. Set $u = -\log p_\nu$ on $\mathbb{C} \setminus K$. Then, u is subharmonic (indeed harmonic) on $\mathbb{C} \setminus K$, and $\lim_{z\to\infty} u(z) = -\infty$. By the continuity principle, if $\zeta \in \partial K$,

$$M \leq \liminf_{\substack{z\to\zeta \\ z\in K}} \log p_\nu(z) = \liminf_{z\to\zeta} \log p_\nu(z) \leq \liminf_{\substack{z\to\zeta \\ z\in\mathbb{C}\setminus K}} \log p_\nu(z),$$

and hence,

$$\limsup_{\substack{z\to\zeta \\ z\in\mathbb{C}\setminus K}} u(z) \le -M.$$

By applying the maximum principle to each component of $\mathbb{C} \setminus K$, we find $u \le -M$ on $\mathbb{C} \setminus K$. It follows that $\log p_\nu \ge M$ on all of $\mathbb{C}$. $\square$

It follows immediately that if p_ν is positive on K then it is positive on all of $\mathbb{C}$. This is important for our purposes, since together with the continuity principle it reveals that if p_ν is continuous and positive on K, and has nice behavior on ∂K, then it is the root function of a multiplicative distance function.

3.2.3 Equilibrium Measures

We now turn our attention to producing multiplicative distance functions from a special class of potentials. These potentials are associated to compact subsets of $\mathbb{C}$, and will allow us to fulfill our goal of producing multiplicative distance functions which measure the complexity of polynomials with respect to certain compact subsets of the complex plane.

As before, let K be a compact subset of $\mathbb{C}$ and let $M(K)$ be the set of Borel probability measures supported on K. There is a topology on $M(K)$ given by the condition that if $\{\nu_n\}$ is a sequence in $M(K)$ then $\nu_n \to \nu$ in $M(K)$ if and only if for any continuous function f supported on K,

$$\int_K f\, d\nu_n \to \int_K f\, d\nu \quad \text{as} \quad n \to \infty.$$

This topology is known as the weak-$*$ topology on $M(K)$, and it can be shown that $M(K)$ is compact in this topology [19].

We define the function $I : M(K) \to [-\infty, \infty)$ by

$$I(\nu) = \int_K \int_K \log|w-\alpha|\, d\nu(w)\, d\nu(\alpha) = \int_K \log p_\nu(\alpha)\, d\nu(\alpha).$$

The quantity $I(\nu)$ is called the *energy* of ν. It can also be shown that I is upper semicontinuous on $M(K)$ in the weak-$*$ topology [12]. It follows that I attains its supremum on $M(K)$, and we define the *capacity* of K to be the quantity

$$c(K) = \exp\left\{ \sup_{\nu \in M(K)} I(\nu) \right\}.$$

Sets which have capacity 0 play a special role in potential theory. These sets are negligible much as sets of measure 0 are negligible in measure theory. In fact, these notions are connected in the sense that a set with capacity zero necessarily has Lebesgue measure zero.

Theorem 3.8. *Suppose K is a compact subset of $\mathbb{C}$ with $c(K) = 0$. Then, $\lambda_2(K) = 0$.*

Proof. Suppose K has positive Lebesgue measure. There exists an $z \in K$, and $r > 0$ such that the closed unit disk $\overline{\Delta}(z, r)$ lies completely in K. Then, for any positive $\rho < r$, let ν_ρ be normalized Lebesgue measure supported on the boundary of the closed disk $\overline{\Delta}(z, \rho)$. By Jensen's formula we find that,

$$\begin{aligned} I(\nu_\rho) &= \int_0^1 \int_0^1 \log \left| \rho e^{2\pi i \theta} - \rho e^{2\pi i \psi} \right| \, d\theta \, d\psi \\ &= \log|\rho| + \int_0^1 \log \max\left\{1, \left|e^{2\pi i \psi}\right|\right\} \, d\psi = \log|\rho|. \end{aligned}$$

Thus, $\nu_\rho \in M(K)$ and $I(\nu_\rho) > -\infty$, and consequently $c(K) > 0$. $\square$

We define an *equilibrium measure* on K to be a probability measure supported on K for which the capacity is attained. That is, a measure $\nu_K \in M(K)$ such that

$$e^{I(\nu_K)} = c(K).$$

By our previous remarks every compact set has at least one equilibrium measure. In fact, if K is a compact subset of $\mathbb{C}$ with positive capacity, then there exists exactly one equilibrium measure (which justifies our notation for equilibrium measures). The proof of this fact would distract us from our main goal of creating multiplicative distance functions from compact sets; the interested reader can find a proof in [12, §3.7] or [8, Appendix 4].

We shall call the potential of ν_K the *equilibrium potential* of K , and use for it the abbreviated notation p_K. Equilibrium potentials give us a way to create multiplicative distance functions associated to sufficiently nice compact subsets of K. Specifically, if K is a connected compact subset of $\mathbb{C}$ such that $\log p_K$ is continuous, then p_K is the root function of a multiplicative distance function which we will denote as P_K.

We will see examples of equilibrium potentials, but first we need more information about them. The following theorem is an important step in the understanding of equilibrium potentials.

Theorem 3.9 (Frostman)**.** *Let K be a compact subset of $\mathbb{C}$ with $c(K) > 0$, and suppose the equilibrium potential p_K is continuous. Then,*

1. *$p_K \geq c(K)$ on $\mathbb{C}$, and*

2. *$p_K = c(K)$ on K.*

Proof. It is easier to work with the logarithmic potential $\log p_K$, and we will show that $\log p_k \geq I(\nu_K)$ on $\mathbb{C}$ and that $\log p_K = I(\nu_K)$ on K. Since $c(K) > 0$, we may assume that $I(\nu_K) > -\infty$.

We will first prove that $\log p_K \leq I(\nu_K)$ on K. For each $n \geq 1$ define

$$U_n = \{z \in K : \log p_\nu(z) > I(\nu) + 1/n\}.$$

We will show that U_n is empty for each $n \geq 1$. Assume then, par contradiction, that U_n is nonempty for some $n \geq 1$. Let $K_n = \overline{U_n}$, and notice that K_n has positive Lebesgue measure. By Theorem 3.8 $c(K_n) > 0$, and hence we may find a measure $\nu \in M(K_n)$ such that $I(\nu) > -\infty$.

Now,

$$I(\nu_K) = \int_K \log p_K(z)\, d\nu_K(z),$$

and hence there exists a $z_1 \in \operatorname{supp} \nu_K$ such that $\log p_K(z_1) \leq I(\nu_K)$. By upper semicontinuity there exists $r_1 > 0$ such that

$$\log p_K(z) < I(\nu_K) + \frac{1}{2n} \quad \text{for all} \quad z \in \overline{\Delta}(z_1; r_1).$$

Notice that $\overline{\Delta}(z_1; r_1) \cap K_n$ is empty. Also, since $z_1 \in \operatorname{supp} \nu_K$, it follows that $\nu_K(\overline{\Delta}(z_1; r_1)) > 0$. Set $a = \nu_K(\overline{\Delta}(z_1; r_1))$ and define the signed measure on K by

$$\sigma = \begin{cases} \nu & \text{on} \quad K_n \\ -\nu_K/a & \text{on} \quad \overline{\Delta}(z_1; r_1) \\ 0 & \text{otherwise} \end{cases}$$

And, for each $t \in (0, a)$ define the measure on K given by

$$\nu_t = \nu_K + t\sigma.$$

It is easy to verify that ν_t is indeed a measure, and since $\sigma(K) = 0$ that it is a probability measure on K.

Now,

$$\begin{aligned} & I(\nu_t) - I(\nu_K) \\ &= \int_K \int_K \log |z - w|\, d\nu_t(w)\, d\nu_t(z) - \int_K \int_K \log |z - w|\, d\nu_K(w)\, d\nu_K(z) \\ &= 2t \int_K \int_K \log |z - w| d\nu_K(z) d\sigma(w) + t^2 \int_K \int_K \log |z - w|\, d\sigma(z)\, d\sigma(w). \end{aligned}$$

The second integral in the latter expression is finite, since $I(\nu) > -\infty$ implies that $I(|\sigma|) > -\infty$. Thus, the second integral is a constant depending only on

σ which we will denote by η. Thus,

$$\begin{aligned}
& I(\nu_t) - I(\nu_K) \\
&= 2t \int_K \log p_K(w)\, d\sigma(w) + t^2\eta \\
&= 2t \int_{K_n} \log p_K(w)\, d\nu(w) - \frac{2t}{a} \int_{\overline{\Delta}(z_1;r_1)} \log p_K(w)\, d\nu_K(w) + t^2\eta \\
&\geq 2t \left\{ \left(I(\nu) + \frac{1}{n} \right) - \left(I(\nu) + \frac{1}{2n} \right) \right\} + t^2\eta \\
&= t\left(\frac{1}{n} + t\eta \right).
\end{aligned}$$

When t is sufficiently small $I(\nu_t) > I(\nu_K)$ contradicting the fact that ν_K is the equilibrium measure of K. We conclude that U_n is empty for all $n \geq 1$, and hence $\log p_K \leq I(\nu_K)$ on K.

We will now prove that $\log p_K \geq I(\nu_K)$ on the support of ν_K. By the minimum principle this implies that $\log p_K \geq I(\nu_K)$ on all of $\mathbb{C}$. From this we will conclude that $\log p_K \geq I(\nu_K)$ on $\mathbb{C}$, and that $\log p_K = I(\nu_K)$ on K.

For each $n \geq 1$, let

$$V_n = \{z \in \operatorname{supp} \nu_K : \log p_K(z) < I(\nu_K)\}.$$

We will show that V_n is empty for each $n \geq 1$, and hence that $\log p_K \geq I(\nu_K)$ on the support of ν_K. Then, by the minimum principle $\log p_K \geq I(\nu_K)$ on all of $\mathbb{C}$, from which part 1 follows immediately.

Assume then that V_n is nonempty for some $n \geq 1$. Choose $z_2 \in V_n$. By upper semicontinuity there exists $r_2 > 0$ such that $\log p_\nu < I(\nu_K) - 1/n$ on $\overline{\Delta}(z_2; r_2)$. Since $z_2 \in \operatorname{supp} \nu_K$ it follows that $\nu_K(\overline{\Delta}(z_2; r_2)) > 0$. Set $b =$

$\nu_K(\overline{\Delta}(z_2; r_2))$. We have already proved that $\log p_K \le I(\nu_K)$ on K and hence,

$$\begin{aligned} I(\nu_K) &= \int_K \log p_K(z)\, d\nu_K(z) \\ &= \int\limits_{\overline{\Delta}(z_2;r_2)} \log p_K(z)\, d\nu_K(z) + \int\limits_{K\setminus\overline{\Delta}(z_2;r_2)} \log p_K(z)\, d\nu_K(z) \\ &\le b\left(I(\nu_K) - \frac{1}{n}\right) + (1-b)I(\nu_K) \\ &< I(\nu_K); \end{aligned}$$

an obvious contradiction. It follows that V_n is empty for each $n \ge 1$, which completes the proof. $\square$

The importance of p_K as a root function for a multiplicative distance function is especially clear when $c(K) = 1$. In this situation $P_K(f) = 1$ exactly when f has all of its roots in K. We see thus that P_K measures the complexity of polynomials by comparing their roots to K. By choosing K to have some arithmetic significance we can create multiplicative distance functions with arithmetic qualities which relate to K. This idea, with a minor modification, allows us to produce multiplicative distance functions which have prescribed behavior with respect to a certain compact subsets of K, even when $c(K) \ne 1$.

Corollary 3.10. *Suppose K is a compact subset of $\mathbb{C}$ with positive capacity, and assume that p_K is continuous. Let Φ be the multiplicative distance function whose root function is given by $\phi(\alpha) = p_K(\alpha) - c(K) + 1$. Then for every $f(x) \in \mathbb{C}[x]$, $\Phi(f) = 1$ if and only if f has all of its roots in K.*

3.3 Green's Functions

The proof of the existence and uniqueness of equilibrium measures is not constructive. For particularly simple compact sets it may be possible to show that a certain measure has maximal energy, however for arbitrary compact sets we must rely on other methods. Using Green's functions we will find that in many instances we can find explicit formulae for equilibrium potentials. That is, Green's functions will allow us to determine Jensen's formulae for multiplicative distance functions formed from equilibrium potentials. This connection is particularly useful since Green's functions have been determined for a wide variety of subdomains of $\mathbb{C}$.

Let $\mathbb{C}_\infty$ be the extended complex plane, and let D be a proper subdomain of $\mathbb{C}_\infty$. A Green's function for D is a map $g_D : D \times D \to (-\infty, \infty]$, such that for each $w \in D$,

1. $g_D(\cdot, w)$ is harmonic on $D \backslash \{w\}$, and bounded outside each neighborhood of w,

2. $g_D(w, w) = \infty$, and as $z \to w$,

$$g_D(z, w) = \begin{cases} \log|z| + O(1) & w = \infty \\ -\log|z - w| + O(1) & w \neq \infty, \end{cases}$$

3. $g_D(z, w) \to 0$ as $z \to \zeta$ for $\zeta \in \partial D$.

Under fairly general conditions on the boundary of D, the Green's function of D exists and is unique. The interested reader can find necessary conditions of the existence and uniqueness of Green's functions in [12, §4.4].

It should be remarked that the standard definition for Green's functions is a bit more general, where condition 3 is expected to hold on $\partial D \setminus E$ where E is a set of capacity 0. However, the definition given here is suitable for the study of multiplicative distance functions, and allows us to avoid certain pathologies.

The next theorem demonstrates the utility of Green's functions in the study of multiplicative distance functions.

Theorem 3.11. *Assume D is a subdomain of C_∞ which contains ∞, and let $K = \mathbb{C}_\infty \setminus D$. If $c(K) > 0$ and p_K is continuous, then*

$$g_D(\alpha, \infty) = \log p_K(\alpha) - \log c(K) \qquad \alpha \in D. \tag{3.3}$$

And consequently,

$$p_K(\alpha) = \begin{cases} c(K) & \alpha \in K \\ c(K) \exp g_D(\alpha, \infty) & \alpha \notin K \end{cases}$$

Proof. By Theorem 3.3, $\log p_K$ is harmonic on $\mathbb{C} \setminus K$. By Lemma 3.5, $\log p_K(\alpha)$ is asymptotic to $\log |\alpha|$ as $\alpha \to \infty$. And, by Frostman's Theorem and the continuity principle for potentials, $\log p_K(\alpha) \to \log c(K)$ as $z \to \zeta$ for $\zeta \in \partial D$. $\square$

Besides providing a way of producing explicit formulae of root functions formed from equilibrium potentials, this theorem also provides a method for computing the capacity of compact sets.

Corollary 3.12. *Let K be a compact subset of $\mathbb{C}$, and let $D = \mathbb{C}_\infty \setminus K$. Then,*

$$\log c(K) = \lim_{\alpha \to \infty} \{\log |\alpha| - g_D(\alpha, \infty)\}.$$

Proof. Clearly from Theorem 3.11,

$$\log c(K) = \log p_\Delta(\alpha) - g_D(\alpha, \infty).$$

The right hand side of this equation is harmonic (indeed constant) on D. By Lemma 3.5 we find,

$$\log c(K) = \lim_{\alpha \to \infty} \{\log |\alpha| - g_D(\alpha, \infty)\}. \qquad \square$$

We conclude that, given a sufficiently nice compact set K, knowledge of the Green's function of $\mathbb{C} \setminus K$ gives us an explicit representation for the root function of the multiplicative distance function P_K. This observation is made more powerful by the following theorem.

Theorem 3.13. *Let K_1 and K_2 be compact subsets of $\mathbb{C}$ with positive capacity. Let $D_1 = \mathbb{C}_\infty \setminus K_1$ and $D_2 = \mathbb{C}_\infty \setminus K_2$. Further suppose that $F : D_1 \to D_2$ is a conformal map of D_1 onto D_2. Then, assuming that g_{D_1} and g_{D_2} exist,*

$$g_{D_2}(F(z), F(w)) = g_{D_1}(z, w) \qquad z, w \in D_1.$$

Before proving this theorem let us look at a couple of examples.

Example 3.3.1. Suppose D is a domain with Green's function g_D, and let $F(z) = 1/z$. Let D' be the image of D under F. The Green's function for D' exists, and is equal to $g_D(1/w, 1/z)$. To see this, notice that:

1. From the axioms of Green's functions, $g_D(1/z, 1/w)$ is a harmonic function of z on $D' \setminus \{w\}$.

2. Clearly, $g_D(1/w, 1/w) = \infty$, and if $w = \infty \in D'$, then as $z \to w$ in D',

$$g_D(1/z, 0) = -\log|1/z| + O(1) = \log|z| + O(1).$$

   If $w = 0 \in D'$, then as $z \to w$ in D',

$$g_D(1/z, \infty) = -\log|z| + O(1) = -\log|z - w| + O(1).$$

   If $w \neq \infty$ and $w \neq 0$, then as $z \to w$ in D',

$$\begin{aligned} g_D(1/z, 1/w) &= -\log|1/z - 1/w| + O(1) \\ &= -\log|z - w| + \lim_{z \to w} \log|zw| + O(1) \\ &= -\log|z - w| + O(1). \end{aligned}$$

3. Since $\partial D'$ is the image of ∂D under F, it follows that $g_D(1/z, 1/w) \to 0$ as $z \to \zeta$ in D' for $\zeta \in \partial D'$.

By the uniqueness of Green's functions we conclude that

$$g_{D'}(z, w) = g_D(1/z, 1/w).$$

Example 3.3.2 (Equilibrium potential for the unit disk)**.** Let Δ be the open unit disk. It can be verified that the Green's function of Δ is given by

$$g_\Delta(z,w) = \log\left|\frac{1-z\overline{w}}{z-w}\right|.$$

The function $z \mapsto 1/z$ is a conformal map sending Δ to $D = \mathbb{C}_\infty \setminus \overline{\Delta}$. And, by Example 3.3.1,

$$g_D(1/z, 1/w) = \log\left|\frac{1-z\overline{w}}{z-w}\right|.$$

And hence, $g_D(\alpha, \infty) = \log|\alpha|$.

Thus, if $f(x) = a_N \prod_{n=1}^N (x-\alpha_n)$, then

$$P_{\overline{\Delta}}(f) = \exp\left\{\int_{\overline{\Delta}} \log|f(w)|\ d\nu_{\overline{\Delta}}(w)\right\} = |a_N| \prod_{n=1}^N \max\{c(\overline{\Delta}), |\alpha_n|\}.$$

From Corollary 3.12 we find that $c(\overline{\Delta}) = 1$, and we conclude that the multiplicative distance function $P_{\overline{\Delta}}$ is the Mahler measure in disguise! We also conclude (if you believe that measures are uniquely determined by their potentials) that the equilibrium measure of the unit disk is simply normalized Lebesgue measure on the unit circle.

Having seen Theorem 3.13 in action, we now turn to its proof. But first, we need a lemma about the positivity of Green's functions.

Lemma 3.14. *Suppose D be a domain with Green's function g_D. Then, $g_D(z,w) > 0$ for all $z, w \in D$.*

Proof. Fix $w \in D$, and let $u(z) = -g_D(z,w)$ for $z \in D$. Then, $u(z)$ is harmonic on $D \setminus \{w\}$, and $u(w) = -\infty$. We conclude that u is subharmonic on D. Furthermore,

$$\limsup_{z\to\zeta} g_D(z,w) = \lim_{z\to\zeta} g_D(z,w) = 0 \quad \text{for} \quad \zeta \in \partial D.$$

By the Maximum Principle we conclude that $u \leq 0$. If $u(z) = 0$ for some $z \in D$, then u is identically zero on D. This is impossible since $u(w) = -\infty$. $\square$

Proof of Theorem 3.13. First, consider the case where $w \neq \infty$ and $F(w) \neq \infty$, and for $z \in D_1 \setminus \{w\}$ define

$$u(z) = g_{D_1}(z, w) - g_{D_2}(F(z), F(w)).$$

It is easily verified that u is harmonic on $D_1 \setminus \{w\}$, and bounded above outside every neighborhood of w. Furthermore,

$$\lim_{z \to w} u(z) = \log \left| \frac{F(z) - F(w)}{z - w} \right| + O(1) = \log |F'(w)| + O(1),$$

And hence, u is bounded above on all of $D_1 \setminus \{w\}$. By Lemma 3.14 $g_{D_2} > 0$, and hence

$$\limsup_{z \ \to \zeta} u(z) \leq \lim_{z \to \zeta} g_{D_1}(z, w) = 0,$$

and by the maximum principle we conclude that $u \leq 0$ on $D_1 \setminus \{w\}$. And thus, when $w \neq \infty$ and $F(w) \neq \infty$,

$$g_{D_1}(z, w) \leq g_{D_2}(F(z), F(w)).$$

By applying the same argument with the conformal map $F^{-1} : D_2 \to D_1$ we find when $w \neq \infty$ and $F(w) \neq \infty$,

$$g_{D_1}(z, w) = g_{D_2}(F(z), F(w)).$$

When $w = \infty$, let $\mathcal{F}$ be the conformal map $z \mapsto 1/z$, and let D_1' be the image of D_1 under this map. From Example 3.3.1 we know that $g_{D'}(z, w) = g_D(1/z, 1/w)$, and from our previous remarks when $F \circ \mathcal{F}(w) \neq \infty$, then

$$g_{D_2}(F \circ \mathcal{F}(z), F \circ \mathcal{F}(w)) = g_{D_1'}(z, w).$$

Consequently,

$$g_{D_1}(z, \infty) = g_{D_1'}(1/z, 0) = g_{D_2}(F(z), F(\infty)).$$

The case when $F(w) = \infty$ is similar, and we leave the details to the reader. $\square$

We end our discussion of Green's function with a corollary to Theorem 3.13 which is important for creating new multiplicative distance functions.

Corollary 3.15. *Let K be a simply connected compact subset of $\mathbb{C}$ with positive capacity and continuous potential. Let $D_1 = \mathbb{C}_\infty \setminus \overline{\Delta}$ and $D_2 = \mathbb{C}_\infty \setminus K$, and suppose $F : D_1 \to D_2$ is a conformal map such that $F(\infty) = \infty$. Then,*

$$p_K(\alpha) = \begin{cases} c(K) & \alpha \in K \\ c(K)\,|F^{-1}(\alpha)| & \alpha \notin K \end{cases}$$

Proof. This follows from Theorem 3.13 and Theorem 3.11, together with that fact that $g_{D_2}(\alpha, \infty) = \log|\alpha|$. □

3.4 Jensen's Formulae for Variants of Mahler Measure

It turns out that the multiplicative distance functions μ_q introduced in Chapter 2 have root functions formed from equilibrium potentials associated to certain ellipses in the complex plane of capacity 1.

Explicitly, let $q \in [0, 1]$ and define the region E_q by

$$E_q = \left\{ x + iy \in \mathbb{C} : \frac{x^2}{(1+q)^2} + \frac{y^2}{(1-q)^2} \le 1 \right\}.$$

Notice that E_0 is the closed unit disk, and E_1 the degenerate ellipse given by the real interval $[-2, 2]$.

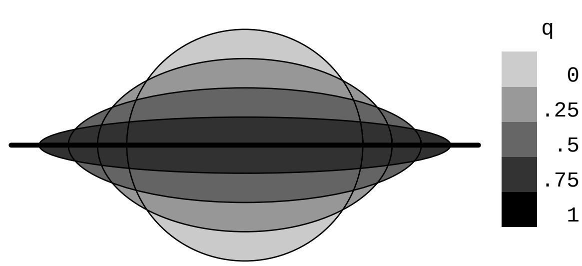

Figure 3.3: E_q for a few values of q

Theorem 3.16. *Let $f(x) \in \mathbb{C}[x]$ and $q \in [0, 1]$, then $\mu_q(f) = P_{E_q}(f)$.*

Before proving this theorem, the result bears some discussion. Recall that, given a polynomial $f(x) \in \mathbb{C}[x]$, $\mu_q(f)$ is equal to the Mahler measure of the Laurent polynomial $f(x + q/x)$. As q varies we are studying the Mahler measure on different families of polynomials, with the family corresponding to $q = 0$ and $q = 1$ being particularly important. This theorem demonstrates a sort of duality whereby, instead of investigating the behavior of the fixed multiplicative distance function μ on the varying collection of polynomials $\mathbb{C}[x + q/x]$, we are also studying the fixed polynomial $f(x)$ under the varying collection of multiplicative distance functions whose root functions are given by the equilibrium potentials of the E_q. Put another way, the study of multiplicative distance functions formed from the compact set E_q gives information about the Mahler measure of Laurent polynomials in $\mathbb{C}[x + q/x]$.

Proof of Theorem 3.16. Consider the map $F(z) = z + q/z$. Clearly $F(\infty) = \infty$, and

$$F(e^{i\theta}) = (1 + q)\cos\theta + (1 - q)i\sin\theta.$$

By hypothesis $q \leq 1$, and thus $F'(z) \neq 0$ when $z \in \mathbb{C}_\infty \setminus \overline{\Delta}$. We conclude that F is a conformal map from $\mathbb{C}_\infty \setminus \overline{\Delta}$ to $\mathbb{C}_\infty \setminus E_q$, with $F(\infty) = \infty$.

Consequently, by Corollary 3.15,

$$p_{E_q}(\alpha) = \begin{cases} c(E_q) & \alpha \in E_q \\ c(E_q)\,|F^{-1}(\alpha)| & \alpha \notin E_q \end{cases}$$

We recover $F^{-1}(\alpha)$ by solving the equation $\alpha = z + q/z$. We fix a branch of the square root, and for each $\alpha \notin E_q$ we find either

$$F^{-1}(\alpha) = \frac{\alpha + \sqrt{\alpha^2 - 4q}}{2} \quad \text{or} \quad F^{-1}(\alpha) = \frac{\alpha - \sqrt{\alpha^2 - 4q}}{2}.$$

It is easily seen that either

$$\left|\frac{\alpha + \sqrt{\alpha^2 - 4q}}{2}\right| > 1 \quad \text{or} \quad \left|\frac{\alpha - \sqrt{\alpha^2 - 4q}}{2}\right| > 1,$$

and thus we can write

$$\left|F^{-1}(\alpha)\right| = \max\left\{1, \left|\frac{\alpha+\sqrt{\alpha^2-4q}}{2}\right|\right\} \max\left\{1, \left|\frac{\alpha-\sqrt{\alpha^2-4q}}{2}\right|\right\}.$$

Also, from Corollary 3.15 we find that the map $\alpha \mapsto |F^{-1}(\alpha)|$ is equal to $g_D(\alpha;\infty)$ where $D = \mathbb{C}_\infty \setminus E_q$. We compute $c(E_q) = 1$ using Corollary 3.12. We have demonstrated that

$$p_{E_q}(\alpha) = \begin{cases} 1 & \alpha \in E_q \\ \max\left\{1, \left|\frac{\alpha+\sqrt{\alpha^2-4q}}{2}\right|\right\} \max\left\{1, \left|\frac{\alpha-\sqrt{\alpha^2-4q}}{2}\right|\right\} & \alpha \notin E_q \end{cases}$$

It remains to show that

$$\phi_q(\alpha) = \max\left\{1, \left|\frac{\alpha+\sqrt{\alpha^2-4q}}{2}\right|\right\} \max\left\{1, \left|\frac{\alpha-\sqrt{\alpha^2-4q}}{2}\right|\right\}$$

is identically 1 on E_q.

This can be verified directly for $q = 1$. For $q < 1$, define functions $\psi_1 : \mathbb{C} \to \mathbb{C}$ and $\psi_2 : \mathbb{C} \to \mathbb{C}$ so that

$$x^2 + \alpha x + q = (x - \psi_1(\alpha))(x - \psi_2(\alpha)).$$

We may assume that ψ_1 and ψ_2 are continuous since the roots of a polynomial depend continuously on the coefficients. Clearly for each $\alpha \in \mathbb{C}$, either

$$\psi_1(\alpha) = \frac{\alpha+\sqrt{\alpha^2-4q}}{2} \quad \text{or} \quad \psi_1(\alpha) = \frac{\alpha-\sqrt{\alpha^2-4q}}{2}.$$

Now, if α is such that $|\psi_1(\alpha)| = 1$, then there exists a $\theta \in [0, 2\pi)$ such that

$$x^2 - \alpha x + q = (x - e^{i\theta})\left(x - \frac{q}{e^{i\theta}}\right)$$

And hence,

$$\alpha = e^{i\theta} + \frac{q}{e^{i\theta}} = F(e^{i\theta}).$$

Since F maps the unit circle to the boundary of E_q, it follows that $|\psi_1(\alpha)| = 1$ only if $\alpha \in \partial E_q$. Similarly $|\psi_2(\alpha)| = 1$ only if $\alpha \in \partial E_q$. Also, $|\psi_1(0)| =$

$|\psi_2(0)| = \sqrt{q} < 1$ and hence both $|\psi_1(\alpha)|$ and $|\psi_2(\alpha)|$ are less than or equal to 1 on E_q. The result follows since

$$\phi_q(\alpha) = \max\{1, |\psi_1(\alpha)|\} \max\{1, |\psi_2(\alpha)|\}. \qquad \square$$

The proof of Theorem 3.16 suggests that the study of multiplicative distance functions formed from certain compact subsets of $\mathbb{C}$ (beyond just the E_q) may yield information about Mahler measure on special collections of polynomials. The next theorem shows that this is indeed the case.

Theorem 3.17. *Let $p(x)$ be a monic polynomial of degree M, and let*

$$F(x) = \frac{p(x)}{x^{M-1}}.$$

If $F'(x)$ does not vanish on $\mathbb{C} \setminus \overline{\Delta}$, then for every $f(x) \in \mathbb{C}[x]$,

$$\mu(f \circ F) = P_K(f),$$

where K is the complement of $F(\mathbb{C} \setminus \overline{\Delta})$.

Proof. The proof of this theorem will be modeled after the proof of Theorem 3.16, though we will have to rely on potential theoretic machinery in lieu of the explicit formulae given for μ_q.

First we compute $\mu(f \circ F)$. By multiplicativity it suffices to determine $\mu(F(x) - \alpha)$ for every $\alpha \in \mathbb{C}$. For each $\alpha \in \mathbb{C}$ there exist M numbers $\psi_1(\alpha), \psi_2(\alpha), \ldots, \psi_M(\alpha)$ such that

$$p(x) - \alpha x^{M-1} = \prod_{m=1}^{M} (x - \psi_m(\alpha)).$$

That is,

$$F(x) - \alpha = x^{-M+1} \prod_{m=1}^{M} (x - \psi_m(\alpha)). \tag{3.4}$$

Since the roots of a polynomial depend continuously on the coefficients we may assume that the maps $\alpha \mapsto \psi_m(\alpha)$ are continuous. Clearly,

$$\mu(F(x) - \alpha) = \prod_{m=1}^{M} \max\{1, |\psi_m(\alpha)|\}.$$

We will complete the proof by showing the right hand side of this equation is exactly $p_K(\alpha)$.

Let $D_1 = \mathbb{C}_\infty \setminus \overline{\Delta}$, and set $D_2 = F(D_1)$. Since F' does not vanish on D_1, F is a conformal mapping from D_1 onto D_2 which maps ∞ to ∞. By Corollary 3.15,

$$p_K(\alpha) = \begin{cases} c(K) & \alpha \in K \\ c(K)|F^{-1}(\alpha)| & \alpha \notin K \end{cases}$$

We may determine F^{-1} from Equation 3.4 by noting that if $\psi_m(\alpha) \neq 0$ then

$$F(\psi_m(\alpha)) = \alpha \quad \text{for} \quad m = 1, 2, \ldots M.$$

If $\alpha \in D_2$ then exactly one of $\psi_1(\alpha), \psi_2(\alpha), \ldots, \psi_M(\alpha)$ is in D_2; otherwise F would fail to be injective on D_1. It follows that on D_2,

$$|F^{-1}(\alpha)| = \prod_{m=1}^{M} \max\{1, |\psi_m(\alpha)|\},$$

and hence,

$$p_K(\alpha) = \begin{cases} c(K) & \alpha \in K \\ c(K) \displaystyle\prod_{m=1}^{M} \max\{1, |\psi_m(\alpha)|\} & \alpha \notin K \end{cases}$$

In order to compute $c(K)$ we turn to Corollary 3.12. By Theorem 3.13,

$$g_K(\alpha, \infty) = \sum_{m=1}^{M} \log \max\{1, |\psi_m(\alpha)|\}.$$

Let b be the coefficient of x^{M-1} in $p(x)$. Then $b - \alpha$ is the coefficient of x^{M-1} in $p(x) - \alpha x^{M-1}$, and thus for each $\alpha \in \mathbb{C}$,

$$b - \alpha = \psi_1(\alpha) + \psi_2(\alpha) + \ldots + \psi_M(\alpha).$$

It follows that $\prod_{m=1}^{M}\max\{1,|\psi_m(\alpha)|\}\sim|\alpha|$ as $\alpha\to\infty$. Thus $g_K(\alpha,\infty)\sim\log|\alpha|$ and hence by Corollary 3.12 we find that $c(K)=1$. We conclude that

$$p_K(\alpha)=\begin{cases}1 & \alpha\in K\\ \displaystyle\prod_{m=1}^{M}\max\{1,|\psi_m(\alpha)|\} & \alpha\notin K\end{cases}$$

All that remains to prove is that

$$p_K(\alpha)=\prod_{m=1}^{M}\max\{1,|\psi_m(\alpha)|\}\quad\text{for every}\quad\alpha\in\mathbb{C}.$$

For each $m=1,2,\ldots,M$, define $u_m:\mathbb{C}\to[1,\infty)$ by $u_m(\alpha)=\max\{1,|\psi_m(\alpha)|\}$. Notice that ψ_m is holomorphic in the region $\{\alpha\in\mathbb{C}:|\psi_m(\alpha)|>1\}$ since in this region it is the inverse of the conformal map F. Thus, from Examples 3.2.2 and 3.2.4, the u_m are subharmonic on $\mathbb{C}$. From the properties of F it is easy to see that

$$\lim_{\substack{\alpha\to\zeta\\ \alpha\in D_2}}|F^{-1}(\alpha)|=1\quad\text{for every}\quad\zeta\in\partial K,$$

and hence,

$$\limsup_{\substack{\alpha\to\zeta\\ \alpha\in K}}u_m(\alpha)\le 1\quad\text{for every}\quad\zeta\in\partial K,$$

It follows from the maximum principle for subharmonic functions that $u_m\le 1$ on the interior of K. We conclude that the map $\alpha\mapsto u_1(\alpha)u_2(\alpha)\cdots u_M(\alpha)$ is identically 1 on K and hence that

$$p_K(\alpha)=\prod_{m=1}^{M}\max\{1,|\psi_m(\alpha)|\}\quad\text{for every}\quad\alpha\in\mathbb{C}.\qquad\square$$

As a corollary to this theorem we give an abstract Jensen's formula for P_K.

Corollary 3.18. *Let F and K be as in the statement of Theorem 3.17, and let $f(x) = a_N \prod_{n=1}^{N}(x - \alpha_n)$. Then,*

$$P_K(f) = |a_N| \prod_{n=1}^{N} \prod_{m=1}^{M} \max\{1, |\psi_m(\alpha_n)|\},$$

where $\psi_1(\alpha), \psi_2(\alpha), \ldots, \psi_M(\alpha)$ are the zeros of $F(x) - \alpha$.

Example 3.4.1. To see Theorem 3.17 in action, let us create a few new multiplicative distance functions. Given $n \geq 1$ let

$$F_n(z) = z + \frac{c_n}{z^n},$$

where c_n is a positive real parameter. Clearly, $F_n'(z)$ vanishes exactly at

$$z = \sqrt[n+1]{c_n n}\, \zeta_{n+1}^m \quad m = 1, 2, \ldots, n-1,$$

where ζ_{n+1} is a primitive $(n+1)$-st root of unity. It follows that when $c_n \leq 1/n$ that all the zeros of F_n' are within the closed unit disk. Thus, in this situation, F_n is a conformal mapping of $\mathbb{C}_\infty \setminus \overline{\Delta}$. In particular the complement of $\mathbb{C}_\infty \setminus \overline{\Delta}$ is some simply connected compact domain in the complex plane. We will denote this compact set K_n in the extremal case where $c_n = 1/n$.

Figure 3.4 shows plots of the boundary of K_n for a few values of n.

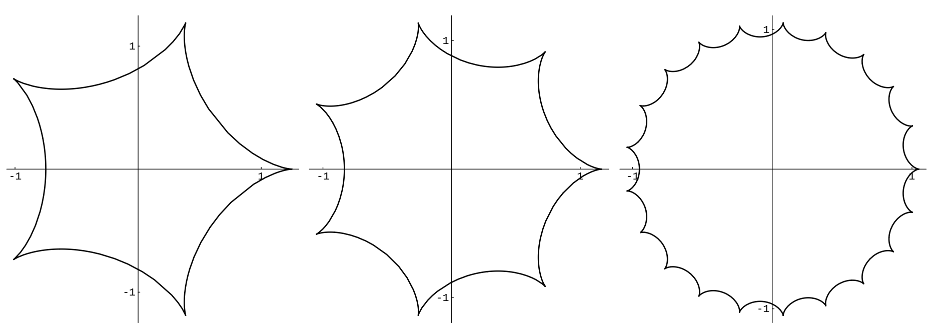

Figure 3.4: Plots of the boundary of K_4, K_6 and K_{20}

By Theorem 3.17 if $f(x)$ is a polynomial, then the Mahler measure of $f \circ F(x)$ is also given by the multiplicative distance function whose root function is given by the equilibrium potential of K_n. A plot of the equilibrium potential of K_6 is given in Figure 3.5.

We remark that $\{P_{K_n}\}$ is another family of multiplicative distance functions having Mahler's measure as a limiting case.

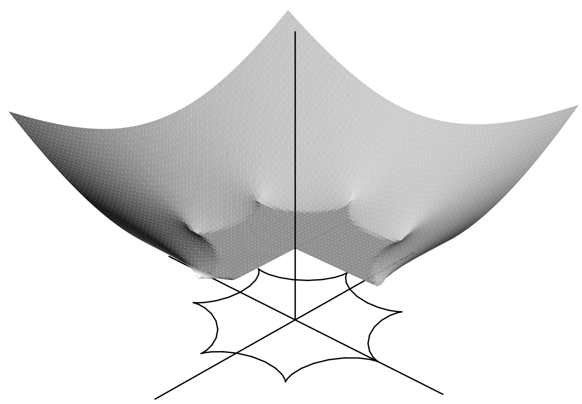

Figure 3.5: A plot of the equilibrium potential of K_6

3.5 Multivariate Multiplicative Distance Functions

Many interesting results have come from the study of the Mahler measure of polynomials in several variables. Given a polynomial $p \in \mathbb{C}[x_1, x_2, \ldots, x_M]$, the Mahler measure of p is defined to be

$$\mu(f) = \exp\left\{\int_0^1 \int_0^1 \cdots \int_0^1 \log\left|p(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, \ldots, e^{2\pi i\theta_M})\right| \, d\theta_1 \, d\theta_2 \cdots d\theta_M\right\}.$$

That is, $\mu(f)$ is the geometric mean of f on the unit torus in $\mathbb{C}^M$. This definition is clearly reliant Jensen's formula.

Any multiplicative distance function possessing a Jensen's formula, can be extended in a like manner. If Φ is a multiplicative distance function such that for some probability measure ν

$$\Phi(f) = \exp\left\{\int_{\mathbb{C}} \log|f(w)|\, d\nu(w)\right\} \quad \text{for every} \quad f \in \mathbb{C}[x],$$

then Φ can be defined for polynomials in several variables. Namely, if p is a polynomial in M variables, then

$$\Phi(p) = \exp\left\{\int_{\mathbb{C}}\int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \log|p(w_1, w_2, \ldots, w_M)|\, d\nu(w_1)\, d\nu(w_2) \cdots d\nu(w_M)\right\}.$$

When ν is a measure with compact support this integral is convergent. It is clearly multiplicative.

We will make no use of multivariate multiplicative distance functions in the sequel — their introduction is only due to the current research interest in Mahler measure of polynomials of several variables.

Part II

Moment Functions of Multiplicative Distance Functions

Chapter 4

Complex Moment Functions

In this chapter we will investigate the distribution of values of multiplicative distance functions. Of particular interest is the volume of star bodies of multiplicative distance functions:

$$\mathrm{vol}(\mathcal{V}_N(\Phi)) = \lambda_{2N+2}\left\{\mathbf{a} \in \mathbb{C}^{N+1} : \Phi(\mathbf{a}) \leq 1\right\},$$

and the cumulative distribution functions of $\widetilde{\Phi}$,

$$h_N(\xi) = h_N(\Phi;\xi) = \lambda_{2N}\left\{\mathbf{b} \in \mathbb{C}^N : \widetilde{\Phi}(\mathbf{b}) \leq \xi\right\}. \tag{4.1}$$

These objects encode information about the range of values of Φ on polynomials in $\mathbb{C}[x]$ of degree N. The homogeneity of Φ establishes a strong connection between these two objects — as we shall see, the volume of $\mathcal{V}_N(\Phi)$ is a special value of the Mellin transform of $h_N(\Phi;\xi)$. We will also be interested in the complex moment function of Φ,

$$H_N(s) = H_N(\Phi;s) = \int_{\mathbb{C}^N} \widetilde{\Phi}(\mathbf{b})^{-2s}\, d\lambda_{2N}(\mathbf{b}) \quad \text{where} \quad s = \sigma + it.$$

As the name suggests, the moments of $\widetilde{\Phi}$ can be recovered from the moment function of Φ; they are special values of $H_N(\Phi;s)$ when s is an integer. The moment function of Φ is closely related to the Mellin transform of $h_N(\Phi;s)$, and we will see that the volume of $\mathcal{V}_N(\Phi)$ is (essentially) one of the moments of $\widetilde{\Phi}$.

The determination of $H_N(\Phi; s)$ will be a central topic of this chapter. This function can be realized as a determinant of a special type of matrix, and from this we will conclude that the volume of $\mathcal{V}_N(\Phi)$ can be represented as the volume of a parallelepiped in a certain Hilbert space determined by Φ. Hilbert space techniques will allow us to further refine $H_N(\Phi; s)$ and the volume of $\mathcal{V}_N(\Phi)$ in terms of a family of orthogonal polynomials associated to Φ. Given an explicit formula for $H_N(\Phi; s)$, a formula for $h_N(\Phi; \xi)$ can be discovered by using the Mellin inversion theorem.

$H_N(\Phi; s)$ and $h_N(\Phi; \xi)$ will be computed for several of the multiplicative distance functions we have introduced in the previous two chapters. Surprisingly, all the examples of $H_N(\Phi; s)$ computed here have a meromorphic continuation to all of $\mathbb{C}$. The location of the zeros and poles of the resulting meromorphic function gives us knowledge about the distribution of values of the particular multiplicative distance functions under consideration. In particular, when $q \in [0, 1]$ we will find that $H_N(\mu_q; s)$ is a rational function of s with poles at small positive and negative integers. Additionally, if q is rational then the coefficients of this rational function are rational numbers times π^N. An immediate consequence of this discovery is that $h_N(\mu_q; \xi)$ is a Laurent polynomial on $[1, \infty)$, and when $q \in \mathbb{Q}$, this polynomial is π^N times a Laurent polynomial with rational coefficients. The appearance of these rational coefficients suggests the presence of a deeper arithmetic phenomenon – one which we will attempt to decipher.

4.1 Volumes of Star Bodies

We will use the method of Chern and Vaaler [3] to write the volume of $\mathcal{V}_N$ as a special value of a Mellin transform of a function determined by Φ.

A brief discussion of the Mellin transform is in order.

4.1.1 Mellin Transforms

The Mellin transform of a function $g : [0, \infty) \to \mathbb{R}$ is given by

$$\widehat{g}(s) = \int_0^\infty \xi^{-s} g(\xi)\, \frac{d\xi}{\xi}, \tag{4.2}$$

where s is a complex variable. The integral defining $\widehat{g}(s)$ converges in a (possibly empty) strip in the complex plane of the form $a < \Re(s) < b$. Where a is a real number determined by the asymptotic behavior of $g(\xi)$ as $\xi \to \infty$ and b is a real number determined by the asymptotic behavior of $g(\xi)$ as $\xi \to 0$. In this strip $\widehat{g}(s)$ is an analytic function. To see this, let Δ be a triangle in $a < \Re(s) < b$, then

$$\begin{aligned}
\int_\Delta \widehat{g}(s)\, ds &= \int_\Delta \left\{ \int_0^\infty \xi^{-2s} g(\xi)\, \frac{d\xi}{\xi} \right\} ds \\
&= \int_{0^+}^\infty g(\xi) \left\{ \int_\Delta \xi^{-2s}\, ds \right\} \frac{d\xi}{\xi} = 0,
\end{aligned}$$

Hence, by Morera's theorem $\widehat{g}(s)$ is analytic in the region of convergence.

4.1.2 Special Values of $\widehat{h_N}(s)$

Theorem 4.1. *Let Φ be a distance function, then the volume of the degree N star body of Φ is given by*

$$\lambda_{2N+2}(\mathcal{V}_N) = 2\pi \widehat{h_N}(2N+2).$$

Proof. The volume of $\mathcal{V}_N$ is given by

$$\lambda_{2N+2}(\mathcal{V}_N) = \int_{\mathbb{C}} \lambda_{2N} \left\{ \mathbf{b} \in \mathbb{C}^N : \Phi(\mathbf{b}, z) \le 1 \right\} d\lambda_2(z)$$

where $(\mathbf{b}, z) = (b_0, \ldots, b_{N-1}, z)$. By the homogeneity of Φ we see

$$\begin{aligned}
\lambda_{2N} \left\{ \mathbf{b} \in \mathbb{C}^N : \Phi(\mathbf{b}, z) \le 1 \right\} &= \lambda_{2N} \left\{ z\mathbf{c} \in \mathbb{C}^N : \Phi(z\mathbf{c}, z) \le 1 \right\} \\
&= |z|^{2N} \lambda_{2N} \left\{ \mathbf{c} \in \mathbb{C}^N : \Phi(\mathbf{c}, 1) \le \frac{1}{|z|} \right\}.
\end{aligned}$$

And thus

$$\begin{aligned}\lambda_{2N+2}(\mathcal{V}_N) &= \int_{\mathbb{C}} |z|^{2N} \lambda_{2N} \left\{ \mathbf{c} \in \mathbb{C}^N : \Phi(\mathbf{c}, 1) \le \frac{1}{|z|} \right\} d\lambda_2(z) \\ &= 2\pi \int_0^\infty r^{2N+1} \lambda_{2N} \left\{ \mathbf{c} \in \mathbb{C}^N : \Phi(\mathbf{c}, 1) \le \frac{1}{r} \right\} dr.\end{aligned}$$

Finally, by setting $\xi = 1/r$ we find

$$\begin{aligned}\lambda_{2N+2}(\mathcal{V}_N) &= 2\pi \int_0^\infty \xi^{-2N-3} \lambda_{2N} \left\{ \mathbf{c} \in \mathbb{C}^N : \Phi(\mathbf{c}, 1) \le \xi \right\} d\xi \\ &= 2\pi \int_0^\infty \xi^{-2N-3} \lambda_{2N} \left\{ \mathbf{c} \in \mathbb{C}^N : \widetilde{\Phi}(\mathbf{c}) \le \xi \right\} d\xi \\ &= 2\pi \widehat{h_N}(2N+2).\end{aligned}$$

□

4.1.3 The Asymptotics of $h_N(\Phi; \xi)$

To gain insight into the nature of h_N we look at the situation geometrically. The set of coefficient vectors of monic polynomials of degree N forms an N dimensional hyperplane in $\mathbb{C}^{N+1}$, and if T is sufficiently large, the dilated star body $T\mathcal{V}_N$ intersects this hyperplane. $h_N(T)$ is the N dimensional Lebesgue measure of the intersection of this hyperplane with $T\mathcal{V}_N$.

From geometric considerations we can determine the asymptotic behavior of $h_N(T)$ as $T \to \infty$ and $T \to 0$.

Proposition 4.2. *Let Φ be a distance function, and let $h_N : [0, \infty) \to [0, \infty)$ be defined as in equation 4.1. Then*

1. *there exists $\epsilon > 0$ such that h_N is identically zero on $[0, \epsilon)$, and*
2. *$h_N(T) = O(T^{2N})$ as $T \to \infty$. Moreover*

$$\lim_{T \to \infty} \frac{h_N(T)}{T^{2N}} = \lambda_{2N}(\mathcal{V}_{N-1}).$$

Proof. Let Δ^{N+1} be the the $N+1$ dimensional unit polydisk centered at the origin. Then, since $\mathcal{V}_N$ is a bounded we can find a positive constant η so that

$$\mathcal{V}_N \subset \eta\Delta^{N+1} \quad \text{and thus} \quad T\mathcal{V}_N \subset T\eta\Delta^{N+1}.$$

Let $B = \{(\mathbf{b}, 1) : \mathbf{b} \in \mathbb{C}^N\}$. B is the hyperplane of coefficient vectors of monic polynomials of degree N. It follows that

$$(B \cap T\mathcal{V}_N) \subset (B \cap T\eta\Delta^{N+1}). \tag{4.3}$$

The set $(B \cap T\eta\Delta^{N+1})$ is an N-dimensional polydisk of radius $T\eta$ if $T\eta \geq 1$, and is empty otherwise. It follows from equation 4.3,

$$h_N(T) \leq \lambda_{2N}(B \cap T\eta\Delta^{N+1}).$$

To prove the first claim in the proposition set $\epsilon = 1/\eta$. If $T < \epsilon$ then $(B \cap T\epsilon\Delta^{N+1})$ is empty. Thus $h_N(T) = 0$ if $T < \epsilon$.

To prove the second claim of the proposition, let $B_{1/T} = \{(\mathbf{b}, 1/T) : \mathbf{b} \in \mathbb{C}^N\}$. Then the set of polynomials with leading coefficient $1/T$ and distance 1 is given by $B_{1/T} \cap \mathcal{V}_N$. Notice that $B_{1/T} = (1/T)B_1$. Thus we find that

$$B_{1/T} \cap \mathcal{V}_N = \frac{1}{T}(B \cap T\mathcal{V}_N).$$

Clearly $(B_{1/T} \cap \mathcal{V}_N) \to \mathcal{V}_{N-1}$ as $T \to \infty$. Thus

$$\begin{aligned}\lambda_{2N}(\mathcal{V}_{N-1}) &= \lim_{T\to\infty} \lambda_{2N}(B_{1/T} \cap \mathcal{V}_N) \\ &= \lim_{T\to\infty} \lambda_{2N}\left(\frac{1}{T}(B \cap T\mathcal{V}_N)\right) = \lim_{T\to\infty} \frac{h_N(T)}{T^{2N}}\end{aligned}$$ □

We have discovered that the volume the degree $N-1$ star body is the coefficient of the leading term of $h_N(T)$:

$$h_N(T) = \lambda_{2N}(\mathcal{V}_{N-1})T^{2N} + o(T^{2N}).$$

The fact that we can recover the volume of $\mathcal{V}_{N-1}$ from h_N should come as no real surprise. After all, h_N is defined by taking the volume of *slices* of $\mathcal{V}_N$, and $\mathcal{V}_{N-1}$ embeds into $\mathcal{V}_N$ as a slice.

4.2 Moment Functions

It shall be convenient to look at $\widehat{h_N}(2s)$ as opposed to $\widehat{h_N}(s)$. Since $\lambda_{2N+2}(\mathcal{V}_N)$ is finite, we know the integral defining $\widehat{h_N}(2s)$ is convergent at $s = N+1$. In fact, $\widehat{h_N}(2s)$ is convergent (and analytic) in the region $\Re(s) > N$. If we regard the integral defining $\widehat{h_N}(2s)$ as a Lebesgue-Stieltjes integral, we may use integration by parts to write

$$\widehat{h_N}(2s) = -\frac{\xi^{-2s} h_N(\xi)}{2s}\bigg|_0^\infty + \frac{1}{2s}\int_0^\infty \xi^{-2s}\, dh_N(\xi). \tag{4.4}$$

It follows from Proposition 4.2 that $h_N(0) = 0$ and $h_N(\xi)$ is dominated by $C\xi^{2N}$ for some constant C. We find that the first term of equation 4.4 vanishes when $\Re(s) > N$. After a change of variables we can write

$$\widehat{h_N}(2s) = \frac{1}{2s}\int_{\mathbb{C}^N} \widetilde{\Phi}(\mathbf{a})^{-2s}\, d\lambda_{2N}(\mathbf{a}), \tag{4.5}$$

and we arrive at the connection between the Mellin transform of $h_N(\xi)$ and the moment function of Φ. It follows from Theorem 4.1 that the volume of $\mathcal{V}_N$ is given by

$$\lambda_{2N+2}(\mathcal{V}_N) = \frac{\pi H_N(N+1)}{N+1}. \tag{4.6}$$

In fact, if $H_N(s)$ has a meromorphic continuation to a neighborhood of $s = N$, then Proposition 4.2(2) implies that the volume of $\mathcal{V}_{N-1}$ is the residue of the pole at $s = N$. We will make no use of this fact, though it is interesting that the residue of $H_N(s)$ at $s = N$ is equal to $H_{N-1}(N)$. This suggests a procedure may exist for determining $H_N(s)$ and the volumes of star bodies via induction on the degree.

Of course, the power of *multiplicative* distance functions lies in the fact that they are multiplicative on $\mathbb{C}[x]$ and the fact that

$$\widetilde{\Phi} : \prod_{n=1}^{N}(x - \alpha_n) \mapsto \prod_{n=1}^{N}\phi(\alpha_n),$$

will be very useful in the evaluation of $H_N(s)$. By changing variables from coefficient vectors to root vectors we will use the multiplicativity to write $H_N(s)$ as an integral which, in many cases, can be evaluated. Even when there is no clear method of evaluation of this integral, we will be able to write $H_N(s)$ as a product N integrals (dependent on s and ϕ) each with domain of integration $\mathbb{C}$. Moreover this integral can be interpreted as an inner product in a certain Hilbert space associated to Φ.

4.2.1 A Change of Variables

We introduce the pivotal change of variables from root vectors to coefficient vectors. Let n be an positive integer less than or equal to N, and let $e_n : \mathbb{C}^N \to \mathbb{C}$ be the nth elementary symmetric function. Explicitly,

$$e_n(\boldsymbol{\alpha}) = (-1)^n \sum_{\mathfrak{t} \in P_n^N} \prod_{\ell=1}^{n} \alpha_{\mathfrak{t}(\ell)},$$

where

$$P_n^N = \{\mathfrak{t} : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, N\} \mid \mathfrak{t}(1) < \mathfrak{t}(2) < \cdots < \mathfrak{t}(n)\}.$$

It is easy to verify then that

$$\prod_{n=1}^{N} (x - \alpha_n) = x^N + \sum_{n=1}^{N} e_n(\boldsymbol{\alpha}) x^{N-n}$$

Let $E_N : \mathbb{C}^N \to \mathbb{C}^N$ be the map whose nth coordinate function is e_n. E_N sends root vectors to monic coefficient vectors. Each monic polynomial is uniquely determined by its roots, and every permutation of the roots leaves $E_N(\boldsymbol{\alpha})$ unchanged. Hence, the degree of the map E_N is $N!$

Also notice that

$$\widetilde{\Phi}(E_N(\boldsymbol{\alpha})) = \prod_{n=1}^{N} \phi(\alpha_n).$$

It is well known (and easy to verify) that the (complex) Jacobian of E_N is given by $|V(\boldsymbol{\alpha})|^2$ where

$$V(\boldsymbol{\alpha}) = \prod_{1 \leq m < n \leq N} (\alpha_n - \alpha_m),$$

And it is clear from this formulation that $|V(\boldsymbol{\alpha})| \neq 0$ for λ_{2N}-almost all points in $\mathbb{C}^N$. Thus the change of variables $\mathbf{a} = E_N(\boldsymbol{\alpha})$ yields

$$H_N(s) = \frac{1}{N!} \int_{\mathbb{C}^N} \left\{ \prod_{n=1}^{N} \phi(\alpha_n)^{-2s} \right\} |V(\boldsymbol{\alpha})|^2 \, d\lambda_{2N}(\boldsymbol{\alpha}). \tag{4.7}$$

This change of variables is useful since it allows us to exploit the multiplicativity of Φ in order to replace an integral over $\mathbb{C}^N$ with a number of integrals over $\mathbb{C}$.

4.2.2 Moment Functions as Determinants

On first inspection, the rewritten form of $H_N(\Phi; s)$ seems more complicated than the original expression. However, this formulation will allow us to use the powerful fact that $V(\boldsymbol{\alpha})$ can be expressed as the famous Vandermonde determinant. This observations together with a bit of combinatorics and Fubini's Theorem will allow us to write $H_N(\Phi; s)$ as a determinant each of whose entries is an integral over $\mathbb{C}$, and these entries can be interpreted as values of an inner product of polynomials in a Hilbert space determined by Φ. Specifically, let ν be the measure supported on the complex plane given by $d\nu(\alpha) = \phi^{-2s}(\alpha) d\lambda_2(\alpha)$, where in this situation we view s as a complex parameter to be chosen later. Then, $L^2(\nu)$ is a Hilbert space with the inner product

$$\langle f | g \rangle = \int_{\mathbb{C}} \phi(\alpha)^{-2s} f(\alpha) \overline{g(\alpha)} \, d\lambda_2(\alpha) \qquad f, g \in L^2(\nu).$$

This inner product induces a norm given by $\mathfrak{N}(f)^2 = \mathfrak{N}(f; s)^2 := \langle f | f \rangle$. When $\Re(s) > N$ then any polynomial in $\mathbb{C}[x]$ with degree less than N is in $L^2(\nu)$.

Now, let $Q = \{Q_n(\alpha) : n = 1, 2, \ldots, N\}$ be a set monic polynomials in $\mathbb{C}[x]$ with $\deg(Q_n) = n - 1$. We will call such a set a *complete* family of polynomials. Each polynomial Q_n is in $L^2(\nu)$ and Q spans a parallelepiped in this Hilbert space. The Gram matrix of Q is defined to be the $N \times N$ matrix, whose j, k entry is given by $\langle Q_j | Q_k \rangle$. This is a symmetric matrix whose j, k entry is dependent on Q_j, Q_k, ϕ and s. The determinant of this matrix can be interpreted as the *volume* of the parallelepiped spanned by Q in the Hilbert space $L^2(\nu)$. Amazingly, the determinant of W_Q is also $H_N(s)$

Theorem 4.3. *Let Q be any complete family of monic polynomials. Then*

$$H_N(s) = \det(W_Q).$$

The power of this theorem is that $H_N(s)$ is valid for *any* complete family of polynomials chosen. Thus, with a smart choice of Q the determinant of W_Q may be quite easy to evaluate. Another powerful feature of this theorem is that the inner product is independent of N. Well, not quite, as we must have $\Re(s) > N$, but viewing s as a parameter to be chosen later the evaluation of the inner products is effectively independent of N.

An obvious choice for Q is a complete family of polynomials which forms an orthogonal set in $L^2(\nu)$. In general the coefficients of such orthogonal polynomials will be dependent on s, and when these coefficients have a meromorphic continuation beyond $\Re(s) > N$, then so will $H_N(s)$.

Corollary 4.4. *Let $\Re(s) > N$, and let Q be a complete family of monic polynomials with*

$$\langle Q_j | Q_k \rangle = \delta_{jk}\, \mathfrak{N}(Q_j; s)^2,$$

where $\delta_{jk} = 1$ if $j = k$ and is 0 otherwise. Then,

$$H_N(s) = \prod_{n=1}^{N} \mathfrak{N}(Q_n; s)^2.$$

To see this corollary in action, suppose Φ is a radial multiplicative distance function. Then,

$$\begin{aligned}\langle \alpha^{j-1} | \alpha^{k-1} \rangle &= \int_{\mathbb{C}} \phi(\alpha)^{-2s} \alpha^{j-1} (\overline{\alpha})^{k-1} \, d\lambda_2(\alpha) \\ &= \int_0^\infty \phi(r)^{-2s} r^{j+k} \frac{dr}{r} \int_0^{2\pi} e^{(j-k)i\theta} \, d\theta.\end{aligned}$$

Parseval's formula implies that

$$\int_0^{2\pi} e^{(j-k)i\theta} \, d\theta = \begin{cases} 0 & \text{if} \quad j \neq k \\ 2\pi & \text{if} \quad j = k \end{cases},$$

and hence $\{1, \alpha, \alpha^2, \ldots, \alpha^{N-1}\}$ is a complete family of orthogonal polynomials in $L^2(\nu)$. It was this fact which made Chern and Vaaler's evaluation of $H_N(\mu; s)$ easy as compared to their evaluation of $F_N(\mu; s)$.

Before proving Theorem 4.3 we need two lemmas.

Lemma 4.5. *Let $I = I(j,k)$ be an $N \times N$ matrix. Then,*

$$\det(I) = \frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) \prod_{n=1}^{N} I(\tau(n), \sigma(n)). \tag{4.8}$$

Proof.

$$\prod_{n=1}^{N} I\left(\tau(n), \sigma(n)\right) = \prod_{n=1}^{N} I(n, \sigma \circ \tau^{-1}(n)).$$

Thus we can write (4.8) as:

$$\begin{aligned}&\frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma \circ \tau^{-1}) \prod_{n=1}^{N} I(n, \sigma \circ \tau^{-1}(n)) \\ = \;& \frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \prod_{n=1}^{N} I(n, \sigma(n)) \\ = \;& \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \prod_{n=1}^{N} I(n, \sigma(n)),\end{aligned}$$

which is the familiar formula for $\det(I)$. $\square$

The next (well-known) lemma gives a convenient representation for $V(\boldsymbol{\alpha})$.

Lemma 4.6. *Let Q be a complete family of monic polynomials. Then,*

$$V(\boldsymbol{\alpha}) = \det(V_Q),$$

where V_Q is the $N \times N$ matrix whose j, k entry is given by $V_Q(j,k) = Q_j(\alpha_k)$.

Proof. This follows easily from the fact that

$$V_Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ * & * & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_N \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{N-1} & \alpha_2^{N-1} & \dots & \alpha_N^{N-1} \end{pmatrix},$$

where $*$ represents entries which are not necessarily 0. The second matrix in this expression is the famous Vandermonde matrix. It is well-known (and easy to verify) that the determinant of this matrix is exactly $V(\boldsymbol{\alpha})$. □

We proceed to the proof of Theorem 4.3

Proof of Theorem 4.3. Recall that

$$H_N(s) = \frac{1}{N!} \int_{\mathbb{C}^N} \left\{ \prod_{n=1}^{N} \phi(\alpha_n)^{-2s} \right\} |V(\boldsymbol{\alpha})|^2 \, d\lambda_{2N}(\boldsymbol{\alpha}).$$

Expanding $\det(V_Q)$ as a sum over S_N we find

$$|V(\boldsymbol{\alpha})|^2 = \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) \prod_{n=1}^{N} Q_{\tau(n)}(\alpha_n) \overline{Q_{\sigma(n)}(\alpha_n)}$$

Substituting this into equation 4.2.2 we can write $H_N(s)$ as

$$\frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) \int_{\mathbb{C}^N} \prod_{n=1}^{N} \phi(\alpha_n)^{-2s} Q_{\tau(n)}(\alpha_n) \overline{Q_{\sigma(n)}(\alpha_n)} \, d\lambda_{2N}(\boldsymbol{\alpha}).$$

Since $H_N(s)$ is convergent when $\Re(s) > N$ we may apply Fubini's theorem. We find

$$H_N(s) = \frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) \prod_{n=1}^{N} \langle Q_{\tau(n)} | Q_{\sigma(n)} \rangle,$$

and the theorem is proved by employing Lemma 4.5. □

4.3 Examples of Moment Functions

4.3.1 Moment Functions of μ_q

Theorem 4.7. *Let N be a positive integer. If $q \in [0,1]$, then $H_N(\mu_q; s)$ analytically continues to the rational function of s given by*

$$H_N(\mu_q; s) = \frac{\pi^N s^N}{N!} \prod_{n=1}^{N} \frac{(1-q^{2n})s + (1+q^{2n})n}{s^2 - n^2}.$$

If $q \in (1, \infty)$ then $H_N(\mu_q; s)$ analytically continues to the meromorphic function of s given by

$$H_N(\mu_q; s) = \frac{q^{-2sN} \pi^N s^N}{N!} \prod_{n=1}^{N} \frac{(-1+q^{2n})s + (1+q^{2n})n}{s^2 - n^2}.$$

There are two values of q for that $H_N(\mu_q; s)$ has been previously computed. The moment functions for $\mu = \mu_0$ was first computed by J. Vaaler and S-J. Chern [3]. In this case we see

$$H_N(\mu; s) = \frac{\pi^N s^N}{N!} \prod_{n=1}^{N} \frac{1}{s-n}.$$

Another special case of Theorem 4.7 was presented in [15] where it was shown that

$$H_N(\mu_1; s) = 2^N \pi^N s^N \prod_{n=1}^{N} \frac{1}{s^2 - n^2}.$$

The proof of this formula given in [15] will motivate our proof of Theorem 4.7.

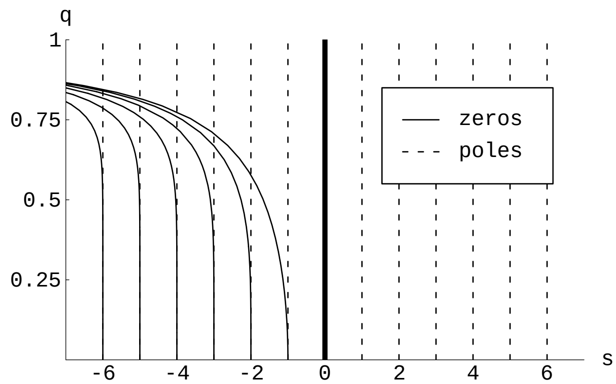

Figure 4.1: The location of the poles and zeros of $H_6(\mu_q; s)$

The set $\{H_N(\mu_q; s) : q \in [0, 1]\}$ forms a curve of rational functions that 'connect' $H_N(\mu; s)$ and $H_N(\mu_1; s)$.

Notice the relationship between $H_N(\mu_q; s)$ and $H_N(\mu_{q^{-1}}; s)$:

$$H_N(\mu_q; s) = q^{-2sN+N(N+1)} H_N(\mu_{q^{-1}}; s). \tag{4.9}$$

When $q > 0$, the poles of $H_N(\mu_q; s)$ are located at both at positive and negative integers, while $H_N(\mu; s)$ has poles only at positive integers. Similarly, each $H_N(\mu_q; s)$ has a zero of multiplicity N at $s = 0$ and, when $q \in (0, 1)$, there are an additional N zeros in the half plane $\Re(s) < 0$. Figure 4.1 shows the location of the poles and zeros for $H_6(\mu_q)$. Notice how the negative zeros of $H_N(\mu_q; s)$ cancel the negative poles of $H_N(\mu_q; s)$ as $q \to 0$. Also note how the negative zeros of $H_N(\mu_q; s)$ approach $-\infty$ as $q \to 1$.

It follows from Equation 4.5 and Theorem 4.7 that

$$\widehat{h_N}(s) = \begin{cases} \dfrac{\pi^N s^{N-1}}{2N!} \displaystyle\prod_{n=1}^{N} \dfrac{(1 - q^{2n})s + (1 + q^{2n})n}{s^2 - n^2} & \text{if} \quad 0 < q \le 1 \\ \dfrac{q^{-2sN} \pi^N s^{N-1}}{2N!} \displaystyle\prod_{n=1}^{N} \dfrac{(-1 + q^{2n})s + (1 + q^{2n})n}{s^2 - n^2} & \text{if} \quad q > 1 \end{cases}.$$

We can recover $h_N(\mu_q; \xi)$ by applying the Mellin inversion formula to this formula. We record this as a corollary of Theorem 4.7.

Corollary 4.8. *Let N be a positive integer and $q \geq 0$, then $h_N(\mu_q; \xi)$ is a Laurent polynomial on the domain $[1, \infty)$ and is identically zero on $[0, 1)$. To be explicit, if $\xi \geq 1$, then*

$$h_N(\mu_q; \xi) = \begin{cases} 2\displaystyle\sum_{n=1}^{N} \left(\frac{\rho_q(n)}{\xi^{2n}} + \sigma_q(n)\xi^{2n} \right) & \text{if} \quad 0 < q \leq 1 \\ 2\, q^{N(N+1)} \displaystyle\sum_{n=1}^{N} \left(\rho_q(n) \frac{q^{2n}}{\xi^{2n}} + \sigma_q(n) \frac{\xi^{2n}}{q^{2n}} \right) & \text{if} \quad q > 1 \end{cases},$$

where

$$\rho_q(n) = \frac{\pi^N n^{N-1}}{2N!} \prod_{\substack{m=1 \\ m \neq n}}^{N} \left(\frac{1}{n-m} - \frac{q^{2m}}{n+m} \right),$$

and

$$\sigma_q(n) = (-1)^N q^{2n} \frac{\pi^N n^{N-1}}{2N!} \prod_{\substack{m=1 \\ m \neq n}}^{N} \left(\frac{q^{2m}}{n-m} - \frac{1}{n+m} \right).$$

Proof. The denominator of $\widehat{h_N}(\mu_q; s)$ is a is a product of distinct linear factors of the form $s - n$. We use partial fraction decomposition to write

$$\widehat{h_N}(\mu_q; s) = \sum_{n=1}^{N} \left(\frac{\rho_q(n)}{s-n} + \frac{\sigma_q(n)}{s+n} \right),$$

where $\rho_q(n) = \operatorname*{Res}_{s=n}(\widehat{h_N}(s))$ and $\sigma_q(n) = \operatorname*{Res}_{s=-n}(\widehat{h_N}(s))$. We compute $\rho_q(n)$:

$$\begin{aligned} (s-n)\widehat{h_N}(s) &= \frac{\pi^N s^{N-1}}{2N!} \left(\frac{(1-q^{2n})s + (1+q^{2n})n}{s+n} \right) \\ &\quad \times \prod_{\substack{m=1 \\ m \neq n}}^{N} \left(\frac{(1-q^{2m})s + (1+q^{2m})m}{s^2 - m^2} \right), \end{aligned}$$

and so

$$\rho_q(n) = \frac{\pi^N n^{N-1}}{2N!} \prod_{\substack{m=1 \\ m \neq n}}^{N} \left(\frac{1}{n-m} - \frac{q^{2m}}{n+m} \right).$$

A similar computation produces the formula for $\sigma_q(n)$ given in the lemma.

The first formula in the corollary follows from the uniqueness of the Mellin transform and the fact that

$$2\int_1^{\infty} \xi^{-2s}\xi^{-2n}\,\frac{d\xi}{\xi} = \frac{1}{s+n}.$$

The second formula follows from Equation 4.9 and the fact that

$$\int_0^{\infty} \xi^{-2s} h_N\left(\xi/q\right)\frac{d\xi}{\xi} = q^{-2s}\int_0^{\infty} \xi^{-2s} h_N(\xi)\,\frac{d\xi}{\xi}. \qquad \square$$

The Proof of Theorem 4.7

We will be content to prove Theorem 4.7 for the case $q \in [0,1]$, and leave the case for $q > 1$ for the reader. We will employ Theorem 4.3 with Q given by $\{1, \alpha, \alpha^2, \ldots, \alpha^{N-1}\}$. A tedious but elementary computation will produce a formula for $\langle \alpha^{J-1} | \alpha^{K-1} \rangle$.

Lemma 4.9. *Let J and K be positive integers such that $J, K < N$, and let $q \in [0,1]$. Then $\langle \alpha^{J-1} | \alpha^{K-1} \rangle$ analytically continues to a rational function of s. Specifically, $\langle \alpha^{J-1} | \alpha^{K-1} \rangle$ equals*

$$2\pi \sum_{n=1}^{N} \left\{ q^{J/2} \begin{bmatrix} J-1 \\ \frac{J+n}{2} \end{bmatrix} \right\} \left\{ q^{K/2} \begin{bmatrix} K-1 \\ \frac{K+n}{2} \end{bmatrix} \frac{s}{2n} \left(\frac{s(q^{-n} - q^n) + n(q^{-n} + q^n)}{s^2 - n^2} \right) \right\},$$

where

$$\begin{bmatrix} J-1 \\ j \end{bmatrix} = \begin{cases} \left[\binom{J-1}{j} - \binom{J-1}{j-1} \right] & \textit{if} \quad j \in \mathbb{Z} \\ 0 & \textit{otherwise} \end{cases}$$

On first inspection, it seems like an unpalatable proposition to find the determinant of W_Q. However, if we define A and B to be the $N \times N$ matrices given by

$$A(J,n) = q^{J/2} \begin{bmatrix} J-1 \\ \frac{J+n}{2} \end{bmatrix}$$

and

$$B(n,K) = \left\{ q^{K/2} \begin{bmatrix} K-1 \\ \frac{K+n}{2} \end{bmatrix} \frac{s}{2n} \left(\frac{s(q^{-n} - q^n) + n(q^{-n} + q^n)}{s^2 - n^2} \right) \right\},$$

then we see that from Lemma 4.9 we see that $W_Q = AB$. Moreover both A and B are triangular triangular matrices (since, for instance if $n > J$ then $\begin{bmatrix} J-1 \\ (J+n)/2 \end{bmatrix} = 0$). We conclude that

$$H_N(\mu_q; s) = (2\pi)^N \det(A)\det(B) = (2\pi)^N \prod_{n=1}^{N} \frac{s}{2n}\left(\frac{s(1-q^{2n}) + n(1+q^{2n})}{s^2-n^2}\right),$$

which simplifies the formula for $H_N(\mu_q; s)$ given in Theorem 4.7.

All that is left to establish is the formulation for $\langle \alpha^{J-1} | \alpha^{K-1}\rangle$ given in Lemma 4.9.

Proof of Lemma 4.9. $\langle \alpha^{J-1} | \alpha^{K-1}\rangle$ is given by

$$= \int_{\mathbb{C}} \max\left\{1, \left|\tfrac{\alpha+\sqrt{\alpha^2-4q}}{2}\right|\right\}^{-2s} \max\left\{1, \left|\tfrac{\alpha-\sqrt{\alpha^2-4q}}{2}\right|\right\}^{-2s} \alpha^{J-1}(\overline{\alpha})^{K-1}\, d\lambda_2(\alpha).$$

By setting $\alpha = \beta + \dfrac{q}{\beta}$ we find

$$\begin{aligned}\langle \alpha^{J-1} | \alpha^{K-1}\rangle = \frac{1}{2}\int_{\mathbb{C}^\times} &\max\{1, |\beta|\}^{-2s} \max\left\{1, \frac{q}{|\beta|}\right\}^{-2s} \\ &\times \left(\beta + \frac{q}{\beta}\right)^{J-1}\left(\overline{\beta} + \frac{q}{\overline{\beta}}\right)^{K-1}\left(\beta - \frac{q}{\beta}\right)\left(\overline{\beta} - \frac{q}{\overline{\beta}}\right)\frac{d\lambda_2(\beta)}{|\beta|^2}.\end{aligned}$$

Writing $\left(\beta + \frac{q}{\beta}\right)^{J-1}$ and $\left(\overline{\beta} + \frac{q}{\overline{\beta}}\right)^{K-1}$ as sums with binomial coefficients $\langle \alpha^{J-1} | \alpha^{K-1}\rangle$ can be written as

$$\begin{aligned}\frac{1}{2}\sum_{j=0}^{J-1}\sum_{k=0}^{K-1}\binom{J-1}{j}\binom{K-1}{k} q^{j+k}\int_{\mathbb{C}^\times} &\max\{1, |\beta|\}^{-2s} \max\left\{1, \frac{q}{|\beta|}\right\}^{-2s} \\ &\times \beta^{J-2j-1}(\overline{\beta})^{K-2k-1}\left(|\beta|^2 + \frac{q^2}{|\beta|^2} - q\left(\frac{\overline{\beta}}{\beta} - \frac{\beta}{\overline{\beta}}\right)\right)\frac{d\lambda_2(\beta)}{|\beta|^2}.\end{aligned}$$

Switching to polar coordinates, $\beta = re^{i\theta}$, and setting

$$\eta(r) = \max\{1, r\}\max\left\{1, \frac{q}{r}\right\}, \tag{4.10}$$

we find

$$\langle \alpha^{J-1} | \alpha^{K-1} \rangle = \frac{1}{2} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \binom{J-1}{j} \binom{K-1}{k} q^{j+k} \int_0^\infty \eta(r)^{-2s} r^{J+K-2(j+k)-2}$$
$$\times \left\{ \int_0^{2\pi} \left(r^2 + \frac{q^2}{r^2} - q(e^{2i\theta} + e^{-2i\theta} \right) e^{(J-K-2(j-k))i\theta}\, d\theta \right\} \frac{dr}{r}. \qquad (4.11)$$

The inner integral in this expression is elementary

$$\int_0^{2\pi} \left(r^2 + \frac{q^2}{r^2} - q\left(e^{2i\theta} + e^{-2i\theta}\right) \right) e^{(J-K-2(j-k))i\theta}\, d\theta$$
$$= \begin{cases} 2\pi \left(r^2 + \frac{q^2}{r^2} \right) & k = j + (K-J)/2 \\ -2\pi q & k = j + 1 + (K-J)/2 \\ -2\pi q & k = j - 1 + (K-J)/2 \end{cases} . \qquad (4.12)$$

We conclude that $\langle \alpha^{J-1} | \alpha^{K-1} \rangle$ is identically zero if $J \not\equiv K \bmod 2$.

From here forward we will assume $J \equiv K \bmod 2$.

The conditions given in equation 4.12 allow us to eliminate one of the summations in equation 4.11. Thus, $\langle \alpha^{J-1} | \alpha^{K-1} \rangle$ can be written as

$$\pi \sum_{j=0}^{J-1} \binom{J-1}{j} \left\{ \binom{K-1}{j+\frac{K-J}{2}} q^{2j+(K-J)/2} \int_0^\infty \eta(r)^{-2s} r^{2J-4j} \frac{dr}{r} \right.$$
$$+ \binom{K-1}{j+\frac{K-J}{2}} q^{2j+(K-J)/2+2} \int_0^\infty \eta(r)^{-2s} r^{2J-4j-4} \frac{dr}{r}$$
$$- \binom{K-1}{j+\frac{K-J}{2}+1} q^{2j+(K-J)/2+2} \int_0^\infty \eta(r)^{-2s} r^{2J-4j-4} \frac{dr}{r}$$
$$\left. - \binom{K-1}{j+\frac{K-J}{2}-1} q^{2j+(K-J)/2} \int_0^\infty \eta(r)^{-2s} r^{2J-4j} \frac{dr}{r} \right\},$$

where as usual we set $\binom{K-1}{k} = 0$ if $k < 0$ or $k > K-1$. Notice that if $K < J$ then some of the terms in the sum are 0. Notice that $\begin{bmatrix} K-1 \\ k \end{bmatrix} = 0$ if $k < 0$ or $k > K$. It follows that

$$\langle \alpha^{J-1} | \alpha^{K-1} \rangle = \pi \sum_{j=0}^{J-1} \binom{J-1}{j} \begin{bmatrix} K-1 \\ j+\frac{K-J}{2} \end{bmatrix} q^{2j+(K-J)/2} \int_0^\infty \eta(r)^{-2s} r^{2J-4j} \frac{dr}{r}$$
$$-\pi \sum_{j=0}^{J-1} \binom{J-1}{j} \begin{bmatrix} K-1 \\ j+\frac{K-J}{2}+1 \end{bmatrix} q^{2j+(K-J)/2+2} \int_0^\infty \eta(r)^{-2s} r^{2J-4j-4} \frac{dr}{r}.$$

Reindexing the second sum by $j \mapsto j-1$ we find

$$\langle \alpha^{J-1} | \alpha^{K-1} \rangle = \pi \sum_{j=0}^{J} \begin{bmatrix} J-1 \\ j \end{bmatrix} \begin{bmatrix} K-1 \\ j+\frac{K-J}{2} \end{bmatrix} q^{2j+(K-J)/2} \int_0^\infty \eta(r)^{-2s} r^{2J-4j} \frac{dr}{r}.$$

The symmetry of $\langle \alpha^{J-1} | \alpha^{K-1} \rangle$ can be seen by setting $j = n + \frac{J}{2}$.

$$\langle \alpha^{J-1} | \alpha^{K-1} \rangle = \pi \sum_{n=-\frac{J}{2}}^{\frac{J}{2}} \begin{bmatrix} J-1 \\ n+\frac{J}{2} \end{bmatrix} \begin{bmatrix} K-1 \\ n+\frac{K}{2} \end{bmatrix} q^{(J+K)/2} q^{2n} \int_0^\infty \eta(r)^{-2s} r^{-4n} \frac{dr}{r}. \tag{4.13}$$

We can take the sum over all half integers, though the terms are only non-zero when $n + J/2$ is an integer. Also, by the definition of $\left[\begin{smallmatrix} J-1 \\ j \end{smallmatrix}\right]$, the term corresponding to $n = 0$ is identically zero. When $n \neq 0$ Equation 4.10 shows

$$\begin{aligned} \int_0^\infty \eta(r)^{-2s} r^{-4n} \frac{dr}{r} &= \int_0^q \left(\frac{q}{r}\right)^{-2s} r^{-4n} \frac{dr}{r} + \int_q^1 r^{-4n} \frac{dr}{r} + \int_1^\infty r^{-2s} r^{-4n} \frac{dr}{r} \\ &= \left. \frac{q^{-2s} r^{2s-4n}}{2s-4n} \right|_0^q + \left. \frac{r^{-4n}}{-4n} \right|_q^1 + \left. \frac{r^{-2s-4n}}{-2s-4n} \right|_1^\infty \\ &= \frac{q^{-4n}}{2s-4n} + \frac{1}{-4n} - \frac{q^{-4n}}{-4n} - \frac{1}{-2s-4n}, \end{aligned} \tag{4.14}$$

when $\Re(s)$ is sufficiently large. By substituting Equation 4.14 into Equation 4.13 we find $\langle \alpha^{J-1} | \alpha^{K-1} \rangle$ equals

$$\pi q^{(J+K)/2} \sum_{n=-\frac{J}{2}}^{\frac{J}{2}} \begin{bmatrix} J-1 \\ n+\frac{J}{2} \end{bmatrix} \begin{bmatrix} K-1 \\ n+\frac{K}{2} \end{bmatrix} \left(\frac{q^{-2n}}{2s-4n} + \frac{q^{-2n}}{4n} - \frac{q^{2n}}{4n} - \frac{q^{2n}}{-2s-4n} \right).$$

Notice that

$$\begin{bmatrix} J-1 \\ -n+\frac{J}{2} \end{bmatrix} = - \begin{bmatrix} J-1 \\ n+\frac{J}{2} \end{bmatrix},$$

and thus we can index the sum defining $\langle \alpha^{J-1} | \alpha^{K-1} \rangle$ by strictly positive half integers. By the substitution $n \mapsto \frac{n}{2}$ and subsequent simplification we arrive at the formulation of $\langle \alpha^{J-1} | \alpha^{K-1} \rangle$ given in the statement of the lemma. □

4.3.2 Moment Functions for Ω_m

Let m be a positive integer and let $\omega_m : \mathbb{C} \to (0,\infty)$ be given by $\omega_m(\alpha) = (1+|\alpha|^m)^{1/m}$, and let Ω_m be the multiplicative distance function given by

$$\Omega_m : a\prod_{n=1}^{N}(x-\alpha_n) \mapsto |a|\prod_{n=1}^{N}\omega_m(\alpha_n).$$

Ω_m is natural and radial. Furthermore

$$\lim_{m\to\infty}\omega_m(\alpha) = \max\{1,|\alpha|\} \quad \text{and thus} \quad \mu = \lim_{m\to\infty}\Omega_m.$$

Since Ω_m is radial we know

$$\begin{aligned} H_N(\Omega_m; s) &= (2\pi)^N \left\{\prod_{n=1}^{N}\int_0^\infty \omega_m(r)^{-2s} r^{2n}\,\frac{dr}{r}\right\} \\ &= (2\pi)^N \left\{\prod_{n=1}^{N}\int_0^\infty (1+r^m)^{-\frac{2s}{m}} r^{2n}\,\frac{dr}{r}\right\}. \end{aligned}$$

The integrals in the previous expression can be evaluated in terms of beta integrals. By the change of variables $t = 1/(1+r^m)$ we find

$$\int_0^\infty (1+r^m)^{-\frac{2s}{m}} r^{2n-1}\,dr = \frac{1}{m}\int_0^1 t^{\frac{2s}{m}-\frac{2n}{m}-1}(1-t)^{\frac{2n}{m}-1}\,dt$$

Recall the identity between the Beta integral and the Gamma function,

$$\int_0^1 t^a(1-t)^b dt = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}. \tag{4.15}$$

Thus we find

$$\int_0^\infty (1+r^m)^{-\frac{2s}{m}} r^{2n-1}\,dr = \frac{\Gamma\left(\frac{2n}{m}\right)\Gamma\left(\frac{2s-2n}{m}\right)}{m\Gamma\left(\frac{2s}{m}\right)}$$

And thus

$$H_N(\Omega_m; s) = (2\pi)^N m^{-N}\prod_{n=1}^{N}\frac{\Gamma\left(\frac{2n}{m}\right)\Gamma\left(\frac{2s-2n}{m}\right)}{\Gamma\left(\frac{2s}{m}\right)}.$$

This expression gives an analytic continuation of $H_N(\Omega_m; s)$ to a meromorphic function on $\mathbb{C}$.

It follows from equation 4.6 that the volume of the degree N star body of Ω_m is given by

$$\begin{aligned}
\lambda_{2N+2}(\mathcal{V}_N(\Omega_m;1)) &= \frac{2^N \pi^{N+1}}{m^N(N+1)} \prod_{n=1}^{N} \frac{\Gamma\left(\frac{2n}{m}\right)\Gamma\left(\frac{2(N+1)-2n}{m}\right)}{\Gamma\left(\frac{2(N+1)}{m}\right)} \\
&= \frac{2^N \pi^{N+1}}{m^N(N+1)} \Gamma(2(N+1)/m)^{-N} \prod_{n=1}^{N} \Gamma\left(\frac{2n}{m}\right)^2 .
\end{aligned}$$

Chapter 5

Real Moment Functions

In this chapter we consider multiplicative distance functions restricted to $\mathbb{R}[x]$. In particular we will investigate an analytic function analogous to $H_N(s)$ which encodes information about the range of values a multiplicative distance function takes on the set of polynomials in $\mathbb{R}[x]$ with degree at most N. The analysis here is a bit more complex since essentially due to the fact that $\mathbb{R}$ is not algebraically complete. However the additional effort is repayed by the fact that information about the range of a multiplicative distance function restricted to polynomials of degree N in $\mathbb{R}[x]$ yields information about the range of that multiplicative distance function on $\mathbb{Z}[x]$. Thus, the facts presented in this chapter, together with more traditional geometry of numbers techniques can yield arithmetic information.

5.1 Star Bodies, Distributions and Moment Functions

The starting point for this chapter is to establish real analogs of $\mathcal{V}_N$, $h_N(\xi)$ and $H_N(s)$ for a multiplicative distance function Φ. Many of the initial ideas an definitions presented here are directly analogous to those in Chapter 4 and hence we will take an expedited journey through many of the initial concepts.

The degree N (real) star body of Φ is defined to be

$$\mathcal{U}_N = \mathcal{U}_N(\Phi) = \left\{ \mathbf{u} \in \mathbb{R}^{N+1} : \Phi(\mathbf{u} \leq 1 \right\}.$$

As before, λ_N is Lebesgue measure on Borel subsets of $\mathbb{R}^N$. Then, by the homogeneity of Φ the volume of the dilated star body $T\mathcal{U}_N$ is given by

$$\lambda_{N+1}(T\mathcal{U}_N) = T^{N+1}\lambda_{N+1}(\mathcal{U}_N) \quad \text{where} \quad T > 0.$$

When T is large the volume of $T\mathcal{U}_N$ gives an estimate for the number of polynomials in $\mathbb{Z}[x]$ with degree at most N and Φ bounded by T. Explicitly

$$\#\left\{\mathbf{u} \in \mathbb{Z}^{N+1}(x) : \Phi(\mathbf{u}) \leq T\right\} \sim T^{N+1}\lambda_{N+1}(\mathcal{U}_N).$$

As in the complex case, the volume of the degree N star body of Φ can be realized as a special value of the Mellin transform of the cumulative distribution function of $\widetilde{\Phi}$.

To be precise, let $f_N : [0,\infty) \to [0,\infty)$ be the cumulative distribution function of Φ given by

$$f_N(\xi) = f_N(\Phi;\xi) = \lambda_N\left\{\mathbf{b} \in \mathbb{R}^{N+1} : \widetilde{\Phi}(\mathbf{b}) \leq \xi\right\}.$$

By essentially the same argument given for Proposition 4.2, the support of f_N is bounded away from 0 and $h_N(T) \sim \lambda_N(\mathcal{U}_{N-1})T^N$.

The following Theorem relates the volume of $\mathcal{U}_N$ with $\widehat{f_N}(s)$.

Theorem 5.1. *Let Φ be a distance function on $\mathbb{R}[x]$, then the volume of the degree N star body of Φ is given by*

$$\lambda_{N+1}(\mathcal{U}_N(\Phi)) = 2\widehat{f_N}(\Phi; N+1).$$

Proof. The proof is essentially identical to the proof of Theorem 4.1. $\square$

Regarding $\widehat{f_N}(s)$ as a Lebesgue-Stieltjes integral, and using integration by parts, it follows that

$$\widehat{f_N}(s) = \frac{1}{s}\int_0^\infty \xi^{-s}\, df_N(\xi) = \frac{1}{s}\int_{\mathbb{R}^N} \widetilde{\Phi}(\mathbf{b})^{-s}\, d\lambda_N(\mathbf{b}).$$

We define the degree N moment function associated to Φ on $\mathbb{R}[x]$ by

$$F_N(s) = F_N(\Phi; s) = \int_{\mathbb{R}^N} \widetilde{\Phi}(\mathbf{b})^{-s} \, d\lambda_N(\mathbf{b}).$$

Much as $H_N(s)$ encodes information about the range of values of Φ on polynomials in $\mathbb{C}[x]$ of degree N, $F_N(s)$ encodes information about the range of values of Φ on polynomials in $\mathbb{R}[x]$ of degree N.

5.2 Moment Functions

The main result of Chapter 4 was that $H_N(s)$ could be represented as the determinant of a matrix whose entries were inner products dependent on s and ϕ. It is our goal of this chapter to produce an analogous result for $F_N(s)$. However, instead of an inner product we will employ a skew-symmetric inner product, and instead of the determinant we will use a related invariant of (antisymmetric) matrices: the Pfaffian.

5.2.1 Skew-Symmetric Inner Products

We define the skew-symmetric inner products $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ by

$$\langle f, g \rangle_{\mathbb{R}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x)^{-s} \phi(y)^{-s} f(x) g(y) \operatorname{sgn}(y - x) \, dx \, dy,$$

and

$$\langle f, g \rangle_{\mathbb{C}} = -2i \int_{\mathbb{C}} \phi(\beta)^{-s} \phi(\overline{\beta})^{-s} \overline{f(\beta)} g(\beta) \operatorname{sgn}(\Im(\beta)) \, d\lambda_2(\beta),$$

where $f, g : \mathbb{C} \to \mathbb{C}$. In fact, these definitions yield a family of skew-symmetric inner products indexed by $s \in \mathbb{C}$. We ignore convergence issues except to note that the integral defining both inner products converge when f and g are polynomials and $\Re(s) > \deg f + \deg g$. Clearly, $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ are dependent on Φ and s – we will suppress such dependencies since we will be working with one multiplicative distance function at a time, and we view s as a complex parameter to be chosen later.

Given a complete family of monic polynomials in $\mathbb{R}[x]$ we may create the $N \times N$ anti-symmetric matrices R_Q and C_Q by specifying that

$$R_Q(j,k) = \langle Q_j, Q_k \rangle_{\mathbb{R}} \qquad \text{and} \qquad C_Q(j,k) = \langle Q_j, Q_k \rangle_{\mathbb{C}}. \tag{5.1}$$

We also define the $N \times N$ anti-symmetric matrix U_Q by specifying that $U_Q = R_Q + C_Q$. Notice the analogy between U_Q and W_Q defined in Section 4.2.2. In other words, we may view U_Q as the antisymmetric analog of a Gram matrix with respect to the skew-symmetric inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{R}} + \langle \cdot, \cdot \rangle_{\mathbb{C}}$. Notice that we are using $\langle \cdot, \cdot \rangle$ for skew-symmetric inner product to distinguish them from the inner products $\langle \cdot | \cdot \rangle$ in the last chapter.

5.2.2 Moment Functions as Pfaffians

An important invariant of anti-symmetric matrices is the Pfaffian. If $N = 2J$, and U is an $N \times N$ anti-symmetric matrix, then the Pfaffian of U is defined to be

$$\mathrm{Pf}(U) = \frac{1}{2^J J!} \sum_{\tau \in S_{2J}} \mathrm{sgn}(\tau) \prod_{j=1}^{J} U(\tau(2j-1), \tau(2j)).$$

Notice the similarity with the expression for the determinant written as a sum over the symmetric group. In fact, the Pfaffian of an antisymmetric matrix can be written as the (signed) square root of the determinant of the matrix. We will not exploit this fact, but mention it only to emphasize the connection between Pfaffians and determinants. It should also be remarked that many of the elementary facts regarding determinants have analogies for the Pfaffian, which can be easily proved by modeling after the corresponding determinantal proof.

The main result of this chapter is that $F_N(s)$ can be written as a Pfaffian (and in some cases as a determinant).

Theorem 5.2. *Let N be an even integer, and let Q be any complete family of monic polynomials in $\mathbb{R}[x]$. Then,*

$$F_N(s) = \mathrm{Pf}(U_Q).$$

The next theorem demonstrates that if ϕ has certain symmetries, we may write $F_N(\Phi; s)$ as a determinant.

Theorem 5.3. *Let $N = 2J$, and let Q be any complete family of monic polynomials in $\mathbb{R}[x]$ such that Q_n is even when $n-1$ is even, and Q_n is odd when $n-1$ is odd. Further suppose the root function of Φ satisfies $\phi(-\beta) = \phi(\beta)$ and $\phi(\overline{\beta}) = \phi(\beta)$ for every $\beta \in \mathbb{C}$. Then,*

$$F_N(s) = \det(A_Q)$$

where A_Q is the $J \times J$ matrix whose j, k entry is given by

$$A_Q(j, k) = U_Q(2j - 1, 2k).$$

As in Theorem 4.3, the power of Theorems 5.2 and 5.3 lie in the fact that they are valid for *any* complete family of monic polynomials which respect the respective hypotheses of the theorems. Thus, a wise choice of Q may make $\mathrm{Pf}(U_Q)$ or $\det(A_Q)$ easy to compute. This begs the question: What is an easy Pfaffian to compute? Certainly, since the Pfaffian is only defined for antisymmetric matrices, diagonal matrixes are out of the picture. A good alternative, however, is given by a block diagonal matrix where each diagonal block consists of a matrix the form rS where

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is easily seen that if U is a $2J \times 2J$ block diagonal matrix consisting of blocks $r_1 S, r_2 S, \ldots, r_J S$ then $\mathrm{Pf}(U) = r_1 r_2 \cdots r_J$. It follows that a sensible choice for Q would be a complete family of monic polynomials for which the Gram matrix with respect to $\langle \cdot, \cdot \rangle$ is in this block diagonal form.

That is, we would like to find a complete family of polynomials Q in $\mathbb{R}[x]$ such that for $1 \leq j, k \leq J$,

$$\langle Q_{2k-1}, Q_{2j} \rangle = -\langle Q_{2j}, Q_{2k-1} \rangle = r_j \delta_{kj} \qquad \langle Q_{2j}, Q_{2k} \rangle = \langle Q_{2j-1}, Q_{2k-1} \rangle = 0. \tag{5.2}$$

In this situation we will say that Q is a complete *skew-orthogonal* family of polynomials. Notice that, since $\langle \cdot, \cdot \rangle$ is dependent on s, then r_j too is a function of s and we will write $r_j = r_j(s)$. Using this we arrive at an analog for Corollary 4.4.

Corollary 5.4. *Let $N = 2J$ and suppose $\Re(s) > N$. Furthermore, let Q be a complete skew-orthogonal family of monic polynomials in $\mathbb{R}[x]$ as in Equation (5.2). Then,*

$$F_N(s) = \prod_{j=1}^{J} r_j(s).$$

5.3 Examples of Moment Functions

Before embarking on the proofs of Theorem 5.2 and Theorem 5.3, we will use Theorem 5.3 to compute $F_N(\mu; s)$ and $F_N(\mu_1; s)$.

First a few remarks on historical approaches to the computation of $F_N(\mu; s)$ are in order. Until recently the only known way to evaluate the integral defining $F_N(\Phi; s)$ was to decompose the domain of integration based on the number of real and complex roots of the associated polynomials, thus replacing one integral with many. A change of variables was then employed to write each of these integrals over the domain of root vectors instead of coefficient vectors. Each of these integrals then had to be decomposed further based on regions where the Jacobian of the change of variables was positive and negative. When the dust settled, one was left with a myriad of integrals, which were tractable for relatively simple multiplicative distance functions. Unfortunately recombining the evaluated integrals into a tractable formula was daunting even in the simplest cases.

S-J. Chern and J. Vaaler were the first to consider $F_N(\mu; s)$ in [3]. Chern and Vaaler demonstrated that $F_N(\mu; s)$ simplifies to a rational function in $\mathbb{Q}(s)$ with a simple product representation via a dispiriting (but admirable) foray into rational function identities. The simplification of $F_N(\mu; s)$ together with qualitative results and conjectures about $F_N(\Phi; s)$ for other multiplicative distance functions (*e.g.* Mahler's measure restricted to reciprocal polynomials) hinted at a larger structure lurking behind the scenes. Moreover, the presence of the Gram matrix in the analogous theory for $H_N(s)$ left us wanting for a similar theory for real moment functions. The scant evidence suggested that there was a mechanism by which $F_N(\Phi; s)$ could be written as a simple product for arbitrary multiplicative distance functions. The fact that $F_N(\Phi; s)$ is a Pfaffian is exactly the mechanism which produces the product formulation.

5.3.1 The Mahler's Measure Case

Before embarking on a proof of Theorems 5.2 and 5.3, we will use Theorem 5.3 to recover the formulation of $F_N(\mu; s)$ given by Chern and Vaaler.

Theorem 5.5 (S.J. Chern, J. Vaaler)**.** *Let $N = 2J$. Then,*

$$F_N(\mu; s) = \mathcal{C}_N \prod_{j=0}^{J-1} \frac{s}{s-(N-2j)}, \qquad \textit{where} \qquad \mathcal{C}_N = 2^N \prod_{j=1}^{J} \left(\frac{2j}{2j+1}\right)^{N-2j}.$$

The following proposition gives the entries in a Gram matrix with respect to the skew-symmetric inner product defined for Mahler's measure. The dependence on Mahler's measure is implicit in the definition of $\langle \cdot, \cdot \rangle$.

Proposition 5.6. *Let $j \leq N$ be an odd positive integer and let $k \leq N$ be an even positive integer. Then,*

$$\langle \gamma^{j-1}, \gamma^{k-1} \rangle = \frac{4}{j(j-k)} \left(\frac{s}{k-s} \right).$$

Sketch of proof. The root function of μ is $\phi(\alpha) = \max\{1, |\alpha|\}$, and hence

$$\begin{aligned}
\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{R}} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max\{1, |x|\}^{-s} \max\{1, |y|\}^{-s} x^{j-1} y^{k-1} \operatorname{sgn}(y-x)\, dx\, dy \\
&= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{y} - \int_{y}^{\infty} \right\} \max\{1, |x|\}^{-s} \max\{1, |y|\}^{-s} x^{j-1} y^{k-1}\, dx\, dy.
\end{aligned}$$

Applying the change of variables $x \mapsto -x$ and $y \mapsto -y$ to one of the integrals in the latter expression we see

$$\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{R}} = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{y} \max\{1, |x|\}^{-s} \max\{1, |y|\}^{-s} x^{j-1} y^{k-1}\, dx\, dy.$$

But this integral is elementary, since we may divide the domain of integration into regions according to where $\max\{1, |x|\}$ and $\max\{1, |y|\}$ are identically one. The integrals converge when $\Re(s) > j + k$. Putting the result into partial fractions form (as a function of s) we find,

$$\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{R}} = \frac{2}{2s - j - k} \left(\frac{4}{j-k} \right) + \frac{4}{j(j+k)} + \frac{2}{s-k} \left(\frac{2k}{j(k-j)} \right).$$

Now,

$$\begin{aligned}
\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{C}} &= -2i \int_{\mathbb{C}} \max\{1, |\beta|\}^{-2s} (\overline{\beta})^{j-1} \beta^{k-1} \operatorname{sgn}(\Im(\beta))\, d\lambda_2(\beta) \\
&= -2i \int_{0}^{\infty} \max\{1, r\}^{-2s} r^{j+k-1}\, dr \times \left\{ \int_0^{\pi} - \int_{\pi}^{2\pi} \right\} e^{(k-j)i\theta}\, d\theta \\
&= -4i \int_{0}^{\infty} \max\{1, r\}^{-2s} r^{j+k-1}\, dr \times \int_0^{\pi} e^{(k-j)i\theta}\, d\theta.
\end{aligned}$$

The two integrals in this expression are elementary, and when $\Re(s) > j + k$, we find

$$\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{C}} = \frac{2}{2s - j - k}\left(\frac{4}{k - j}\right) + \frac{8}{(k - j)(k + j)}$$

Notice that the first term in $\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{C}}$ exactly cancels the first term in our expression for $\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{R}}$. That is,

$$\begin{aligned}\langle \gamma^{j-1}, \gamma^{k-1} \rangle &= \frac{4}{j(j + k)} + \frac{2}{s - k}\left(\frac{2k}{j(k - j)}\right) + \frac{8}{(k - j)(k + j)} \\ &= \frac{4}{j(j - k)}\left(\frac{s}{k - s}\right). \qquad \square\end{aligned}$$

Proof of Theorem 5.5. We will use Theorem 5.3, which is justified since the root function of μ clearly satisfies the hypotheses of the theorem. For simplicity we will set $Q = \{1, \gamma, \gamma^2, \ldots, \gamma^{N-1}\}$.. Thus we need to determine $\langle \gamma^{j-1}, \gamma^{k-1} \rangle$ where $j \leq N$ is odd and $k \leq N$ is even. From Proposition 5.6 may write $F_N(\mu; s) = \det(A)$ where

$$A(j, k) = U(2j - 1, 2k) = \frac{s}{s - 2k}\left(\frac{4}{2j - 1}\right)\left(\frac{1}{2k - 2j + 1}\right).$$

Each entry in the kth column is multiplied by $s/(s - 2k)$ and each entry in the jth row is multiplied by $4/(2j - 1)$, and thus

$$\det(A) = \det(B) \prod_{j=1}^{J} \frac{s}{s - 2j}\left(\frac{4}{2j - 1}\right), \tag{5.3}$$

where B is the $J \times J$ matrix given by $B(j, k) = (2k - 2j + 1)^{-1}$. The determinant of B is of a special form known by T. Muir as a *double alternant* [11, §353]. From Muir, the determinant of B is given by

$$\det(B) = (-1)^{\binom{J}{2}} \left\{ \prod_{1 \leq j < k \leq J} (2k - 2j)^2 \right\} \Big/ \left\{ \prod_{j=1}^{J} \prod_{k=1}^{J} (2k - 2j + 1) \right\}. \tag{5.4}$$

The denominator of which is

$$\begin{aligned}\prod_{j=1}^{J}\prod_{k=1}^{J}(2k-2j+1) &= (-1)^{\binom{J}{2}}\prod_{1\le j<k\le J}(2(k-j)+1)(2(k-j)-1)\\ &= (-1)^{\binom{J}{2}}\prod_{1\le j<k\le J}(2(k-j)+1)\prod_{1\le j<k\le J-1}(2(k-j)+1)\\ &= (-1)^{\binom{J}{2}}\prod_{1\le j<k\le J}(2(k-j)+1)^2\left\{\prod_{j=1}^{J}(2(J-j)+1)\right\}.\end{aligned}$$

Substituting this into (5.4) we find

$$\left\{\prod_{j=1}^{J}\frac{4}{2j-1}\right\}\det(B) = 4^J\prod_{j=1}^{J}\left(\frac{2j}{2j+1}\right)^{2J-2j}.$$

And thus,

$$F_N(\mu;s) = 2^N\prod_{j=1}^{J}\left(\frac{2j}{2j+1}\right)^{N-2j}\prod_{j=1}^{J}\frac{s}{s-2j},$$

which after reindexing yields the formula for $F_N(\mu;s)$ given in the statement of the theorem. □

5.3.2 The Reciprocal Mahler's Measure Case

Theorem 5.7. *Let $N = 2J$. Then,*

$$F_N(\mu_1;s) = v_N\prod_{j=0}^{J-1}\frac{s^2}{s^2-(N-2j)^2}, \quad \textit{where} \quad v_N = \frac{2^N}{N!}\prod_{n=1}^{N}\left(\frac{2n}{2n-1}\right)^{N+1-n}.$$

We now compute the entries in a Gram matrix with respect to the skew-symmetric inner product defined for μ_1. As in the Mahler's measure case the dependence on μ_1 is implicit in the definition of $\langle\cdot,\cdot\rangle$.

Proposition 5.8. *Let $j \le N$ be an odd positive integer and let $k \le N$ be an even positive integer. Then*

$$\langle\gamma^{j-1},\gamma^{k-1}\rangle = \sum_{n=1}^{k}\sum_{m=1}^{j}\begin{bmatrix} j-1\\ \frac{j-m}{2}\end{bmatrix}\begin{bmatrix} k-1\\ \frac{k-n}{2}\end{bmatrix}\frac{n}{m}\left(\frac{16}{n^2-m^2}\right)\left(\frac{s^2}{s^2-n^2}\right),$$

where

$$\begin{bmatrix} M \\ m \end{bmatrix} = \binom{M}{m} - \binom{M}{m-1}$$

if m is an integer, and 0 otherwise.

Notice that $\binom{M}{m}$ are exactly the variations of binomial coefficients which appear in the calculation of $H_N(\mu_1; s)$. This is certainly not a coincidence, though the full relationship between $H_N(s)$ and $F_N(s)$ is not currently well understood.

A few facts about these binomial-like coefficients are in order before proving Proposition 5.8.

Lemma 5.9. *Let j and k be positive integers. Then,*

1. $\left(x + \frac{1}{x}\right)^{j-1}\left(x - \frac{1}{x}\right) = \sum_{m=1}^{j} \begin{bmatrix} j-1 \\ m \end{bmatrix} x^{j-2m}.$

2. *If j is odd, then* $\frac{2^j}{j} = \sum_{m=0}^{j} \begin{bmatrix} j-1 \\ \frac{j-m}{2} \end{bmatrix} \frac{2}{m}.$

3. *If j is odd and k is even, then*

$$\frac{2^{j+k}}{j(j+k)} = \sum_{n=1}^{k}\sum_{m=1}^{j} \begin{bmatrix} k-1 \\ \frac{k-n}{2} \end{bmatrix} \begin{bmatrix} j-1 \\ \frac{j-m}{2} \end{bmatrix} \frac{n}{m}\left(\frac{4m}{n^2-m^2}\right).$$

Proof. To prove (1) we use the Binomial Theorem to expand

$$(x + 1/x)^{j-1}(x - 1/x)$$

and collect together terms with like powers of x.

Now let ω be a path in the complex plane that does not pass through $z = 0$ and consider the path integral

$$\int_\omega \left(z + \frac{1}{z}\right)^{j-1}\left(1 - \frac{1}{z^2}\right) dz. \tag{5.5}$$

If j is odd then (1) implies that the integrand consists of even powers of x and hence the integral depends only on the end points of ω.

To prove (2) notice that

$$\frac{2^j}{j} = \frac{1}{2}\int_{-2}^{2} x^{j-1}\,dx = \frac{1}{2}\int_{\omega}\left(z+\frac{1}{z}\right)^{j-1}\left(1-\frac{1}{z^2}\right)dz,$$

where the second equality follows from the change of variables $x \mapsto z + 1/z$, and ω is any path in the complex plane starting at $z = -1$ ending at $z = 1$ and not passing through $z = 0$. Using (1) and the Fundamental Theorem of Calculus we find

$$\frac{2^j}{j} = \sum_{m=0}^{j}\begin{bmatrix} j-1 \\ m \end{bmatrix}\frac{1}{j-2m}.$$

The change of variables $j \mapsto j - 2m$ together with the fact that

$$\begin{bmatrix} j-1 \\ \frac{j+m}{2} \end{bmatrix} = -\begin{bmatrix} j-1 \\ \frac{j-m}{2} \end{bmatrix}$$

allows us to simplify this to the form given in (2).

To prove (3) we notice that

$$\begin{aligned}
\frac{2^{j+k}}{j(j+k)} &= \frac{1}{2}\int_{-2}^{2} y^{k-1}\int_{-2}^{y} x^{j-1}\,dx\,dy \\
&= \frac{1}{2}\sum_{m=0}^{j}\begin{bmatrix} j-1 \\ m \end{bmatrix}\int_{-2}^{2} y^{k-1}\int_{-2}^{\phi_+(y)} x^{j-2m-1}\,dx\,dy,
\end{aligned}$$

where the second equality stems from the change of variables $x \mapsto x + 1/x$, and $\phi_+(y) = (y + \sqrt{y^2-4})/2$. Again we use the fact that j is odd to conclude that the resulting integral is path independent. Assuming that k is even we may evaluate the inner integral and simplify to find

$$\begin{aligned}
\frac{2^{j+k}}{j(j+k)} &= \frac{1}{2}\sum_{m=0}^{j}\begin{bmatrix} j-1 \\ m \end{bmatrix}\frac{1}{j-2m}\int_{-2}^{2} y^{k-1}\phi_+(y)^{j-2m}\,dy \\
&= \frac{1}{2}\sum_{n=0}^{k}\sum_{m=0}^{j}\begin{bmatrix} j-1 \\ m \end{bmatrix}\begin{bmatrix} k-1 \\ n \end{bmatrix}\frac{1}{j-2m}\int_{-1}^{1} y^{j+k-2m-2n-1}\,dy \\
&= \frac{1}{2}\sum_{n=0}^{k}\sum_{m=0}^{j}\begin{bmatrix} j-1 \\ m \end{bmatrix}\begin{bmatrix} k-1 \\ n \end{bmatrix}\frac{1}{j-2m}\times\frac{2}{j-2m+k-2n}.
\end{aligned}$$

Reindexing by $m \mapsto j-2m$ and $n \mapsto k-2n$ we find

$$\begin{aligned}
\frac{2^{j+k}}{j(j+k)} &= \sum_{n=-k}^{k}\sum_{m=-j}^{j}\begin{bmatrix} j-1 \\ m \end{bmatrix}\begin{bmatrix} k-1 \\ n \end{bmatrix}\frac{1}{m}\times\frac{1}{m+n} \\
&= \sum_{n=0}^{k}\sum_{m=0}^{j}\begin{bmatrix} j-1 \\ m \end{bmatrix}\begin{bmatrix} k-1 \\ n \end{bmatrix}\left\{\frac{2}{m}\left(\frac{1}{n+m}+\frac{1}{n-m}\right)\right\},
\end{aligned}$$

where the last equality is a consequence of the fact that $\begin{bmatrix} j-1 \\ \frac{j+m}{2} \end{bmatrix} = -\begin{bmatrix} j-1 \\ \frac{j-m}{2} \end{bmatrix}$. Simplifying this expression we discover (3). $\square$

Proof of Proposition 5.8. First we evaluate $\langle \gamma^{j-1}, \gamma^{k-1}\rangle_{\mathbb{R}}$. It is easily seen that

$$\langle \gamma^{j-1}, \gamma^{k-1}\rangle_{\mathbb{R}} = 2\int_{\infty}^{\infty}\int_{-\infty}^{y}\phi_1(x)^{-s}\phi_1(y)^{-s}\,x^{j-1}y^{k-1}\,dx\,dy. \tag{5.6}$$

Let α be a real number and define

$$\phi_-(\alpha) = \frac{\alpha-\sqrt{\alpha^2-4}}{2} \qquad \text{and} \qquad \phi_+(\alpha) = \frac{\alpha+\sqrt{\alpha^2-4}}{2},$$

for any fixed branch of the square root. It is easily seen that if $\alpha < 2$ then $\phi_1(\alpha) = -\phi_-(\alpha)$ and if $\alpha > 2$ then $\phi_1(\alpha) = \phi_+(\alpha)$. Moreover when $2 \le \alpha \le 2$ $\phi(\alpha) = 1$. Using this we may write the inner integral in (5.6) as

$$\mathcal{F}(y) = \int_{-\infty}^{y}\phi_1(x)\,x^{j-1}\,dx = \begin{cases} \displaystyle\int_{-\infty}^{y}(-\phi_-(x))^{-s}\,x^{j-1}\,dx & y \le 2 \\ \displaystyle\int_{-2}^{y}x^{j-1}\,dx + \mathcal{F}(-2) & -2 < y \le 2 \\ \displaystyle\int_{2}^{y}\phi_+(x)^{-s}\,x^{j-1}\,dx + \mathcal{F}(2) & 2 < y. \end{cases}$$

When $\Re(s) > j$ each of the integrals in this expression converges. When $y \le 2$ the integral defining $\mathcal{F}(y)$ is elementary by noticing that after the change of variables $x \mapsto x + 1/x$,

$$\begin{aligned} \mathcal{F}(y) &= \int_{-\infty}^{\phi_-(y)} (-x)^{-s} \left(x + \frac{1}{x}\right)^{j-1} \left(x - \frac{1}{x}\right) \frac{dx}{x} \\ &= \sum_{m=0}^{j} \begin{bmatrix} j-1 \\ m \end{bmatrix} \int_{-\infty}^{\phi_-(y)} (-x)^{-s}\, x^{j-2m-1}\, dx, \end{aligned}$$

where the last equality follows from Lemma 5.9. When $y > 2$ we may use the same change of variables to write

$$\mathcal{F}(y) = \sum_{m=0}^{j} \begin{bmatrix} j-1 \\ m \end{bmatrix} \int_{1}^{\phi_+(y)} x^{-s} x^{j-2m-1}\, dx.$$

Again, the integrals in this expression converge when $\Re(s) > j$. Evaluating these elementary integrals, and noting that j is odd we see

$$\mathcal{F}(y) = \begin{cases} \displaystyle\sum_{m=0}^{j} \begin{bmatrix} j-1 \\ m \end{bmatrix} \frac{(-\phi_-(y))^{-s}\phi_-(y)^{j-2m}}{-s+j-2m} & y \le 2 \\[2ex] \displaystyle\frac{y^j + 2^j}{j} - \sum_{m=0}^{j} \begin{bmatrix} j-1 \\ m \end{bmatrix} \frac{1}{-s+j-2m} & -2 < y \le 2 \\[2ex] \displaystyle\frac{2^{j+1}}{j} + \sum_{m=0}^{j} \begin{bmatrix} j-1 \\ m \end{bmatrix} \frac{\phi_+(y)^{-s+j-2m} - 2}{-s+j-2m} & 2 < y. \end{cases} \tag{5.7}$$

Now $\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{R}}$ is given by

$$2\left\{ \int_{\infty}^{-2} (-\phi_-(y))^{-s} y^{k-1} \mathcal{F}(y)\, dy + \int_{-2}^{2} y^{k-1}\mathcal{F}(y)\, dy + \int_{2}^{\infty} \phi_+(y)^{-s} y^{k-1} \mathcal{F}(y)\, dy \right\}. \tag{5.8}$$

The first integral of which is given by

$$\sum_{m=0}^{j} \begin{bmatrix} j-1 \\ m \end{bmatrix} \int_{-\infty}^{-2} (-\phi_-(y))^{-2s} \phi(y)^{j-2m} y^{k-1}\, dy.$$

The change of variables $y \mapsto y + 1/y$ together with Lemma 5.9 yields

$$\sum_{m=0}^{j}\sum_{n=0}^{k}\begin{bmatrix} j-1 \\ m \end{bmatrix}\begin{bmatrix} k-1 \\ n \end{bmatrix}\int_{-\infty}^{-1}(-y)^{-2s}y^{j-2m+k-2n-1}\,dy.$$

The integrals in this expression are, of course, elementary and converge when $\Re(s) > \max\{j,k\}$. Substituting (5.7) into Equation 5.8 we find the second integral in (5.8) is elementary, and after the change of variables $y \mapsto y + 1/y$, the third integral becomes

$$\sum_{m=0}^{j}\sum_{n=0}^{k}\begin{bmatrix} j-1 \\ m \end{bmatrix}\begin{bmatrix} k-1 \\ n \end{bmatrix}\int_{1}^{\infty} y^{-2s+j-2m+k-2n-1}\,dy.$$

Again, the integrals in this expression are elementary and converge when $\Re(s) > \max\{j,k\}$. Evaluating all these integrals together with the fact that k is even we find that

$$\begin{aligned}
\langle\gamma^{j-1},\gamma^{k-1}\rangle_{\mathbb{R}} &= \frac{2^{j+k+2}}{j(j+k)} + \frac{2^{j+2}}{j}\sum_{n=0}^{k}\begin{bmatrix} k-1 \\ n \end{bmatrix}\frac{-1}{-s+k-2n} \\
&+4\sum_{n=0}^{k}\sum_{m=0}^{j}\begin{bmatrix} k-1 \\ n \end{bmatrix}\begin{bmatrix} j-1 \\ m \end{bmatrix}\frac{1}{2m-2n+k-j}\times\frac{1}{s-k+2n} \\
&-8\sum_{n=0}^{k}\sum_{m=0}^{j}\begin{bmatrix} k-1 \\ n \end{bmatrix}\begin{bmatrix} j-1 \\ m \end{bmatrix}\frac{1}{2m-2n+k-j}\times\frac{1}{2s-k-j+2n+2m}.
\end{aligned} \tag{5.9}$$

We now turn our attention to $\langle\gamma^{j-1},\gamma^{k-1}\rangle_{\mathbb{C}}$.

$$\langle\gamma^{j-1},\gamma^{k-1}\rangle_{\mathbb{C}} = -2i\int_{\mathbb{C}}\phi_1(\beta)^{-2s}(\overline{\beta})^{j-1}\beta^{k-1}\operatorname{sgn}(\Im(\beta))\,d\lambda_(\beta).$$

After the change of variables $\beta \mapsto \beta + 1/\beta$ we may rewrite $\langle\gamma^{j-1},\gamma^{k-1}\rangle_{\mathbb{C}}$ as

$$-i\int_{\mathbb{C}}\max\{|\beta|,\left|\beta^{-1}\right|\}^{-2s}\left(\overline{\beta}+\frac{1}{\overline{\beta}}\right)^{j-1}\left(\beta+\frac{1}{\beta}\right)^{k-1}\left|\beta-\frac{1}{\beta}\right|^{2} \times\operatorname{sgn}\left(\Im\left(\beta+\frac{1}{\beta}\right)\right)\frac{d\lambda_2(\beta)}{|\beta|^2}.$$

Clearly this is invariant under the change of variables $\beta \mapsto 1/\beta$, and thus we may replace the domain of integration with $\mathbb{C}\setminus\overline{\Delta}$ where $\overline{\Delta}$ is the closed unit disk. In this domain, $\operatorname{sgn}(\Im(\beta + 1/\beta)) = 1$ if β in the open upper half plane, and is equal to -1 if β is in the open lower half plane. After an easy simplification we may use these facts to rewrite $\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{C}}$ as

$$-4i \int\limits_{H^+ \setminus \overline{\Delta}} |\beta|^{-2s} \left(\overline{\beta} + \frac{1}{\overline{\beta}}\right)^{j-1} \left(\beta + \frac{1}{\beta}\right)^{k-1} \left(\overline{\beta} - \frac{1}{\overline{\beta}}\right) \left(\beta - \frac{1}{\beta}\right) \frac{d\lambda_2(\beta)}{|\beta|^2},$$

where H^+ is the open upper half plane. Employing Lemma 5.9 we may rewrite this as

$$-4i \sum_{m=0}^{j} \sum_{n=0}^{k} \begin{bmatrix} j-1 \\ m \end{bmatrix} \begin{bmatrix} k-1 \\ n \end{bmatrix} \int\limits_{H^+ \setminus \overline{\Delta}} |\beta|^{-2s} (\overline{\beta})^{j-2m-1} \beta^{k-2n-1} \, d\lambda_2 \beta.$$

Switching to polar coordinates this becomes

$$-4i \sum_{m=0}^{j} \sum_{n=0}^{k} \begin{bmatrix} j-1 \\ m \end{bmatrix} \begin{bmatrix} k-1 \\ n \end{bmatrix} \int_0^{\pi} e^{(2m-2n+k-j)i\theta} \, d\theta \int_1^{\infty} r^{-2s+k+j-2n-2m-1} \, dr.$$

Of course, these integrals are elementary and we finally can write $\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{C}}$ as

$$8 \sum_{n=0}^{k} \sum_{m=0}^{j} \begin{bmatrix} k-1 \\ n \end{bmatrix} \begin{bmatrix} j-1 \\ m \end{bmatrix} \frac{1}{2m-2n+k-j} \times \frac{1}{2s-k-j+2n+2m}.$$

Notice that this exactly cancels the third term in Equation 5.9 and hence,

$$\begin{aligned} \langle \gamma^{j-1}, \gamma^{k-1} \rangle \quad = \quad & \frac{2^{j+k+2}}{j(j+k)} + \frac{2^{j+2}}{j} \sum_{n=0}^{k} \begin{bmatrix} k-1 \\ n \end{bmatrix} \frac{-1}{-s+k-2n} \\ & + 4 \sum_{n=0}^{k} \sum_{m=0}^{j} \begin{bmatrix} k-1 \\ n \end{bmatrix} \begin{bmatrix} j-1 \\ m \end{bmatrix} \frac{1}{2m-2n+k-j} \times \frac{1}{s-k+2n}. \end{aligned}$$

Reindexing the sums in this expression by $m \mapsto j - 2m$ and $n \mapsto k - 2n$ this becomes

$$\frac{2^{j+k+2}}{j(j+k)} + \frac{2^{j+2}}{j} \sum_{n=-k}^{k} \begin{bmatrix} k-1 \\ \frac{k-n}{2} \end{bmatrix} \frac{1}{s-n} + 4 \sum_{n=-k}^{k} \sum_{m=-j}^{j} \begin{bmatrix} k-1 \\ \frac{k-n}{2} \end{bmatrix} \begin{bmatrix} j-1 \\ \frac{j-m}{2} \end{bmatrix} \frac{1}{n-m} \times \frac{1}{s-n}.$$

By noticing that $\left[\begin{smallmatrix} k-1 \\ (k+n)/2 \end{smallmatrix}\right] = -\left[\begin{smallmatrix} k-1 \\ (k-n)/2 \end{smallmatrix}\right]$ we may reindex these sums over only non-negative integers to find

$$4\left\{\frac{2^{j+k}}{j(j+k)} + \sum_{n=1}^{k} \begin{bmatrix} k-1 \\ \frac{k-n}{2} \end{bmatrix} \frac{2n}{s^2-n^2}\left(\frac{2^j}{j} + \sum_{m=1}^{j} \begin{bmatrix} j-1 \\ \frac{j-m}{2} \end{bmatrix} \frac{2m}{n^2-m^2}\right)\right\}.$$

Replacing $2^j/j$ with the expression given in Lemma 5.9 together with the fact that $\frac{2m}{n^2-m^2} + \frac{2}{m} = \frac{2n^2}{m(n^2-m^2)}$, we find $\langle \gamma^{j-1}, \gamma^{k-1}\rangle$ can be written as

$$4\left\{\frac{2^{j+k}}{j(j+k)} + \sum_{n=1}^{k}\sum_{m=1}^{j} \begin{bmatrix} j-1 \\ \frac{j-m}{2} \end{bmatrix} \begin{bmatrix} k-1 \\ \frac{k-n}{2} \end{bmatrix} \frac{n^2}{s^2-n^2} \times \frac{4n}{m(n^2-m^2)}\right\}.$$

And, since $\frac{n^2}{s^2-n^2} = \frac{s^2}{s^2-n^2} - 1$,

$$\begin{aligned} \langle \gamma^{j-1}, \gamma^{k-1}\rangle \;\; &= \;\; \sum_{n=1}^{k}\sum_{m=1}^{j} \begin{bmatrix} j-1 \\ \frac{j-m}{2} \end{bmatrix} \begin{bmatrix} k-1 \\ \frac{k-n}{2} \end{bmatrix} \frac{n}{m}\left(\frac{16}{n^2-m^2}\right)\left(\frac{s^2}{s^2-n^2}\right) \\ &\quad + \left\{\frac{2^{j+k+2}}{j(j+k)} - \sum_{n=1}^{k}\sum_{m=1}^{j} \begin{bmatrix} j-1 \\ \frac{j-m}{2} \end{bmatrix} \begin{bmatrix} k-1 \\ \frac{k-n}{2} \end{bmatrix} \frac{n}{m}\left(\frac{16}{n^2-m^2}\right)\right\}. \end{aligned}$$

From Lemma 5.9 we find that the term in braces vanishes and we arrive at the formulation for $\langle \gamma^{j-1}, \gamma^{k-1}\rangle$ given in the statement of the proposition. $\square$

Proof of Theorem 5.7. Clearly μ_1 satisfies the hypotheses of Theorem 5.3. Thus, Since $\left[\begin{smallmatrix} M \\ m \end{smallmatrix}\right] = 0$ if $m > M$ or $m < 0$, we can write $F_N(\mu_1; s) = \det(A)$, where A is the $J \times J$ matrix whose j, k entry is given by

$$A(j,k) = 16\sum_{n=1}^{N}\sum_{m=1}^{N} \begin{bmatrix} 2j-2 \\ j-\frac{m+1}{2} \end{bmatrix} \begin{bmatrix} 2k-1 \\ k-\frac{n}{2} \end{bmatrix} \frac{n}{m}\left(\frac{1}{n^2-m^2}\right)\left(\frac{s^2}{s^2-n^2}\right).$$

By noting that the double sum in this expression is zero unless n is even and m is odd, we may reindex by $m \mapsto 2m-1$ and $n \mapsto 2n$ to write $A(j,k)$

$$\sum_{n=1}^{J} \underbrace{\begin{bmatrix} 2k-1 \\ k-n \end{bmatrix}\left(\frac{s^2}{s^2-(2n)^2}\right)}_{B(k,n)} \sum_{m=1}^{J} \underbrace{\begin{bmatrix} 2j-2 \\ j-m \end{bmatrix}}_{C(j,m)} \underbrace{\frac{2n}{(2m-1)}\left(\frac{16}{(2n)^2-(2m-1)^2}\right)}_{D(m,n)}.$$

Defining the $J \times J$ matrices B, C and D via this equation it is easily verified that $A = BCD$,and hence $F_N(\mu_1; s) = \det(B)\det(C)\det(D)$. This is useful since both B and C are lower triangular matrices (since, for instance if $m > j$ then $\begin{bmatrix} 2j-2 \\ j-m \end{bmatrix} = 0$). Computing the diagonal entries of B and C we find,

$$F_N(\mu_1; s) = \det(D) \prod_{j=0}^{J-1} \frac{s^2}{s^2 - (N-2j)^2}.$$

And, since D is a matrix of rational numbers we have proven Theorem 5.7 except for the identity of the constant v_n. Clearly,

$$\det(D) = \left\{ 4^N \prod_{j=1}^{J} \frac{2j}{2j-1} \right\} \det(E), \quad \text{where} \quad E(m,n) = \frac{1}{(2n)^2 - (2m-1)^2}.$$

Notice that E is a double alternant, and we may use [11, §353] to determine an expression for $\det(E)$.

However, we will use a result of S. DiPippo and E. Howe to recover the expression for v_n given in the statement of the theorem. To do this, notice that $\det(D) = \lim_{s \to \infty} F_N(\mu_1; s)$. And thus,

$$\det(D) = \int_\Omega d\lambda_N(\mathbf{a}), \tag{5.10}$$

where

$$\Omega = \left\{ \mathbf{a} \in \mathbb{R}^N : x^N + \sum_{n=1}^{N} a_n x^{N-n} \text{ has all roots in } [-2, 2] \right\}.$$

The integral in Equation 5.10 was evaluated by DiPippo and Howe in [4, §2]. Using this result we find that $\det(D) = v_N$ where v_N is given as in the statement of the theorem. $\square$

5.4 The Proof of Theorem 4.3

5.4.1 A Change of Variables

First, let $N = L + 2M$ and define the map $E_{L,M} : \mathbb{R}^L \times \mathbb{C}^M \to \mathbb{R}^N$ by $E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathbf{a}$, where

$$x^N + \sum_{n=1}^{N} a_{N-n} x^n = \prod_{\ell=1}^{L} (x - \alpha_\ell) \prod_{m=1}^{M} (x - \overline{\beta}_m)(x - \beta_m).$$

That is, $E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the monic coefficient vector of the polynomial whose roots are $\alpha_1, \ldots \alpha_L$ and $\overline{\beta_1}, \beta_1, \ldots, \overline{\beta_M}, \beta_M$. Now, let

$$\mathcal{D}_{L,M} = \left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}^L \times \mathbb{C}^M : \Im(\beta_m) \neq 0, m = 1, 2, \ldots, M \right\}.$$

It is clear that $\mathbb{R}^N$ is the disjoint union,

$$\mathbb{R}^N = \bigcup_{L+2M=N} E_{L,M}(\mathcal{D}_{L,M}).$$

Next we need a lemma which gives the Jacobian of $E_{L,M}$.

Lemma 5.10. *Let L and M be positive integers such that $L + 2M = N$, and let*

$$V(\gamma_1, \ldots, \gamma_N) = \prod_{1 \leq m < n \leq N} (\gamma_n - \gamma_m). \tag{5.11}$$

Then,

$$\operatorname{Jac}\left(E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta})\right) = 2^M \left| V(\alpha_1, \ldots, \alpha_L, \overline{\beta_1}, \beta_1, \ldots, \overline{\beta_M}, \beta_M) \right|.$$

Proof. First we compute the Jacobian matrix of $E_{L,M}$. If $0 \leq j \leq N-1$, the jth coordinate function of $E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is given by

$$e_{N-j}(\alpha_1, \ldots, \alpha_L, \overline{\beta_1}, \beta_1, \ldots, \overline{\beta_M}, \beta_M),$$

where e_n is the nth elementary symmetric function. The following abbreviations will be useful. For $1 \leq n \leq N$ set,

$$e_n = e_n(\boldsymbol{\alpha}, \boldsymbol{\beta}) = e_n(\alpha_1, \ldots, \alpha_L, \overline{\beta_1}, \beta_1, \ldots, \overline{\beta_M}, \beta_M),$$

and we understand that $e_0 = 1$. We also specify that if $1 \leq \ell < L$ then $e_{n,\ell} = e_{n,\ell}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the n-the elementary symmetric function in all of our variables except α_ℓ. Similarly if $1 \leq m < M$ then we define $e'_{n,m} = e'_{n,m}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to be the n-th elementary symmetric function in all of our variables except β_m and $\overline{\beta_m}$.

Using these definitions it is easy to see that for $1 \leq \ell \leq L$,

$$\frac{\partial e_{N-j}}{\partial \alpha_\ell} = e_{N-j-1,\ell}.$$

Setting $\beta_m = x_m + iy_m$ we may compute the partial derivatives of e_{N-j} with respect to x_m and y_m. Clearly, e_{N-j} consists of four types of monomials: those which contain β_m but not $\overline{\beta_m}$, those which contain $\overline{\beta_m}$ but not β_m, those which contain both β_m and $\overline{\beta_m}$, and those which contain neither β_m nor $\overline{\beta_m}$.

$$e_{N-j} = (\beta_m + \overline{\beta_m})\, e'_{N-j-1,m} + \beta_m \overline{\beta_m}\,' e_{N-j-2,m} + e'_{N-j,m}.$$

And thus,

$$e_{N-j} = 2x_m\, e'_{N-j-1,m} + (x_m^2 + y_m^2)\, e'_{N-j-2,m} + e'_{N-j,m}.$$

From which it follows that

$$\frac{\partial e_{N-j}}{\partial x_m} = 2e'_{N-j-1,m} + 2x_m e'_{N-j-2,m} \qquad \text{and} \qquad \frac{\partial e_{N-j}}{\partial y_m} = 2y_m e'_{N-j-2,m}.$$

Since we are interested in the determinant of the Jacobian matrix, we may reindex by $j \mapsto N - j$ to write the Jacobian of $E_{L,M}$ as the determinant of J where the j-th row of J is given by

$$\begin{aligned}(e_{j-1,1} \quad e_{j-1,2} \quad \cdots \quad e_{j-1,L} \quad 2e'_{j-1,1} + 2x_1\, e'_{j-2,1} \quad 2y_1\, e'_{j-2,1} \\ \cdots \quad 2e'_{j-1,M} + 2x_M\, e'_{j-2,M} \quad 2y_M\, e'_{j-2,M}).\end{aligned}$$

Now let C be the $L \times L$ diagonal matrix whose jth diagonal entry is given by $(-1)^{j-1}$, and let

$$B = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

And let A be the $N \times N$ block diagonal matrix

$$A = \begin{pmatrix} C & & & & \\ & B & & & \\ & & B & & \\ & & & \ddots & \\ & & & & B \end{pmatrix}.$$

We then set $J' = AJ$ and notice that the j-th row of J' is given by

$$\begin{aligned} &(-1)^{j-1}(e_{j-1,1} \quad e_{j-1,2} \quad \cdots \quad e_{j-1,L} \\ &\qquad e'_{j-1,1} + \beta_1\, e'_{j-2,1} \qquad e'_{j-1,1} + \overline{\beta}_1\, e'_{j-2,1} \\ &\qquad\quad \cdots \quad e'_{j-1,M} + \beta_M\, e'_{j-2,M} \qquad e'_{j-1,M} + \overline{\beta}_M\, e'_{j-2,M}), \end{aligned} \tag{5.12}$$

and it is easily seen that $|\det(J)| = 2^M |\det(J')|$.

Now, let $1 \leq \ell \leq L$ and define $f_\ell(x)$ to be the polynomial

$$f_\ell(x) = \prod_{\substack{k=1 \\ k\neq\ell}}^{L} (x - \alpha_k) \prod_{m=1}^{M} (x - \beta_m)(x - \overline{\beta_m}) = x^{N-1} + \sum_{n=1}^{N} (-1)^n e_{n,\ell}\, x^{N-1-n}.$$

Similarly, for $1 \leq m \leq M$ define g_m and $\tilde{g}_m$ by

$$\begin{aligned} g_m(x) &= \prod_{\ell=1}^{L} (x - \alpha_\ell) \left\{ (x - \beta_m) \prod_{\substack{k=1 \\ k\neq m}}^{M} (x - \beta_k)(x - \overline{\beta_k}) \right\} \\ &= (x - \beta_m) \left(x^{N-2} + \sum_{n=1}^{N-2} (-1)^n e'_{n,m}\, x^{N-2-n} \right) \\ &= x^{N-1} + \sum_{n=1}^{N-1} (-1)^n (e'_{n,m} + \beta_m e'_{n-1,m}) x^{N-1-n}, \end{aligned}$$

and

$$\begin{aligned} \tilde{g}_m(x) &= \prod_{\ell=1}^{L} (x - \alpha_\ell) \left\{ (x - \overline{\beta_m}) \prod_{\substack{k=1 \\ k\neq m}}^{M} (x - \beta_k)(x - \overline{\beta_k}) \right\} \\ &= x^{N-1} + \sum_{n=1}^{N-1} (-1)^n (e'_{n,m} + \overline{\beta_m} e'_{n-1,m}) x^{N-1-n}. \end{aligned}$$

We now use the definitions of f_ℓ, g_m and $\tilde{g}_m$ to create some useful orthogonality relations. By construction, $f_\ell(\beta_m) = f_\ell(\overline{\beta_m}) = 0$ for all $1 \leq \ell \leq L$ and $1 \leq m \leq M$, and

$$f_\ell(\alpha_k) = \begin{cases} \displaystyle\prod_{j \neq \ell} (\alpha_\ell - \alpha_j) \prod_{m=1}^{M} (\alpha_\ell - \beta_m)(\alpha_\ell - \overline{\beta_m}) & \text{if } k = \ell \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $g_m(\alpha_\ell) = \tilde{g_m}(\alpha_\ell) g_m(\beta_m) = \tilde{g}_m(\overline{\beta}) = 0$ for all $1 \leq \ell \leq L$ and $1 \leq m \leq M$, and

$$g_m(\overline{\beta_k}) = \begin{cases} \displaystyle\prod_{\ell=1}^{L} (\overline{\beta_m} - \alpha_\ell) \left((\overline{\beta_m} - \beta_m) \prod_{j \neq m} (\overline{\beta_m} - \beta_j)(\overline{\beta_m} - \overline{\beta_j}) \right) & \text{if } k = m \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{g}_m(\beta_k) = \begin{cases} \displaystyle\prod_{\ell=1}^{L} (\beta_m - \alpha_\ell) \left((\beta_m - \overline{\beta_m}) \prod_{j \neq m} (\beta_m - \beta_j)(\beta_m - \overline{\beta_j}) \right) & \text{if } k = m \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $D = D(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be the $N \times N$ matrix given by

$$D = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{N-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_L & \alpha_L^2 & \cdots & \alpha_L^{N-1} \\ 1 & \overline{\beta_1} & \overline{\beta_1}^2 & \cdots & \overline{\beta_1}^{N-1} \\ 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{\beta_M} & \overline{\beta_M}^2 & \cdots & \overline{\beta_M}^{M-1} \\ 1 & \beta_M & \beta_M^2 & \cdots & \beta_M^{M-1} \end{pmatrix}.$$

That is D is the $N \times N$ Vandermonde matrix in the variables

$$\alpha_1, \alpha_2, \ldots, \alpha_L, \overline{\beta_1}, \beta_1, \ldots, \overline{\beta_M}, \beta_M.$$

And it is well known that

$$\det(D) = V(\alpha_1, \alpha_2, \ldots, \alpha_L, \overline{\beta_1}, \beta_1, \ldots, \overline{\beta_M}, \beta_M) =: V(\boldsymbol{\alpha}, \boldsymbol{\beta}).$$

Now, from the definitions of D and J' (Equation 5.12) we find

$$DJ' = \begin{pmatrix} f_1(\alpha_1) & \cdots & f_L(\alpha_1) & g_1(\alpha_1) & \tilde{g}_1(\alpha_1) & \cdots & g_M(\alpha_1) & \tilde{g}_M(\alpha_1) \\ f_1(\alpha_2) & \cdots & f_L(\alpha_2) & g_1(\alpha_2) & \tilde{g}_1(\alpha_2) & \cdots & g_M(\alpha_2) & \tilde{g}_M(\alpha_2) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_1(\alpha_L) & \cdots & f_L(\alpha_L) & g_1(\alpha_L) & \tilde{g}_1(\alpha_L) & \cdots & g_M(\alpha_L) & \tilde{g}_M(\alpha_L) \\ f_1(\overline{\beta_1}) & \cdots & f_L(\overline{\beta_1}) & g_1(\overline{\beta_1}) & \tilde{g}_1(\overline{\beta_1}) & \cdots & g_M(\overline{\beta_1}) & \tilde{g}_M(\overline{\beta_1}) \\ f_1(\beta_1) & \cdots & f_L(\beta_1) & g_1(\beta_1) & \tilde{g}_1(\beta_1) & \cdots & g_M(\beta_1) & \tilde{g}_M(\beta_1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_1(\overline{\beta_M}) & \cdots & f_L(\overline{\beta_M}) & g_1(\overline{\beta_M}) & \tilde{g}_1(\overline{\beta_M}) & \cdots & g_M(\overline{\beta_M}) & \tilde{g}_M(\overline{\beta_M}) \\ f_1(\beta_M) & \cdots & f_L(\beta_M) & g_1(\beta_M) & \tilde{g}_1(\beta_M) & \cdots & g_M(\beta_M) & \tilde{g}_M(\beta_M) \end{pmatrix}$$

But from the orthogonality relations we see that this is in fact a diagonal matrix, and moreover the determinant of DJ' is given by

$$\det(DJ') = \prod_{\ell=1}^{L} f_\ell(\alpha_\ell) \prod_{m=1}^{M} g_1(\overline{\beta_1})\tilde{g}_1(\beta_1) = V(\boldsymbol{\alpha}, \boldsymbol{\beta})^2.$$

But this implies that $|\det(J')| = |V(\boldsymbol{\alpha}, \boldsymbol{\beta})|$, and hence

$$\operatorname{Jac}(E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta})) = |\det J| = 2^M |\det J'| = 2^M |V(\boldsymbol{\alpha}, \boldsymbol{\beta})|. \qquad \square$$

This change of variables is useful, since it exploits the multiplicativity Φ. That is,

$$\widetilde{\Phi}\,(E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta}))^{-s} = \prod_{l=1}^{L} \phi(\alpha_l)^{-s} \prod_{m=1}^{M} \phi(\overline{\beta_m})^{-s} \phi(\beta_m)^{-s},$$

and hence $F_N(\Phi; s)$ is given by

$$\sum_{L+2M=N} \frac{1}{L!M!} \int_{\mathcal{D}_{L,M}} \prod_{l=1}^{L} \phi(\alpha_l)^{-s} \prod_{m=1}^{M} \phi(\overline{\beta_m})^{-s} \phi(\beta_m)^{-s} |V(\boldsymbol{\alpha}, \boldsymbol{\beta})| d(\lambda_L \times \lambda_{2M})(\boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{5.13}$$

Our difficulties are apparent in this formulation of $F_N(\Phi; s)$: Firstly, the sum over integers L and M such that $L + 2M = N$ leads to a combinatorial nightmare. Secondly, the appearance of the absolute value of $V(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is difficult to manage since we must first determine where $V(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is positive and negative.

5.4.2 Combinatorial Preliminaries

We introduce some notation which will prove convenient for dealing with some of the combinatorial results presented in this manuscript. This notation will allow us to concisely state several combinatorial lemmas. Many of these lemmas are technical in nature with proofs which, on first inspection, seem overtly formal. So that the formal aspects of the proofs do not obfuscate the main ideas behind the proof of Theorem 5.2 we will try to illuminate the main ideas behind the more technical proofs.

Let N be a positive integer. For each $K \leq N$ we set

$$P_K^N = \{\mathfrak{t} : \{1, 2, \ldots, K\} \to \{1, 2, \ldots, N\} \mid \mathfrak{t}(1) < \mathfrak{t}(2) < \cdots < \mathfrak{t}(K)\}.$$

The image of $\mathfrak{t}$ will be denoted by $\mathrm{im}(\mathfrak{t})$. Associated to each $\mathfrak{t} \in P_K^N$ there exists a unique $\mathfrak{t}' \in P_{N-K}^N$ such that $\mathrm{im}(\mathfrak{t}) \cap \mathrm{im}(\mathfrak{t}') = \emptyset$. We also associate to $\mathfrak{t}$ the subset of the symmetric group S_N given by

$$S_N(\mathfrak{t}) = \{\tau \in S_N : \tau(j) \in \mathrm{im}(\mathfrak{t}) \text{ for } j = 1, 2, \ldots, K\}$$

For $\tau \in S_N(\mathfrak{t})$ define $\sigma_\tau \in S_K$ and $\pi_\tau \in S_{N-K}$ by specifying that

$$\begin{array}{rcll} \sigma_\tau(j) & = & \mathfrak{t}^{-1}(\tau(j)) & \quad j = 1, 2, \ldots, K \\ \pi_\tau(\ell) & = & \mathfrak{t}'^{-1}(\tau(K + \ell)) & \quad \ell = 1, 2, \ldots, N - K. \end{array}$$

From this we define the *signature* of $\mathfrak{t}$ to be

$$\mathrm{sgn}(\mathfrak{t}) = \frac{\mathrm{sgn}(\tau)}{\mathrm{sgn}(\pi_\tau)\,\mathrm{sgn}(\sigma_\tau)}.$$

To see that $\operatorname{sgn}(\mathfrak{t})$ is well defined, let τ' be another element of $S_N(\mathfrak{t})$ and notice that for each $j \in \{1, 2, \ldots, K\}$ there exists $j' \in \{1, 2, \ldots, K\}$ such that

$$\tau(j) = \mathfrak{t}(\sigma_\tau(j)) = \mathfrak{t}(\sigma_{\tau'}(j')) = \tau'(j'), \quad \text{and thus} \quad \tau^{-1}\circ\tau'(j') = \sigma_\tau^{-1}\circ\sigma_{\tau'}(j').$$

Similarly, for each $\ell \in \{1, 2, \ldots, N-K\}$ there exist $\ell' \in \{1, 2, \ldots, N-K\}$ such that

$$\tau(K+\ell) = \mathfrak{t}'(\pi(\ell)) = \mathfrak{t}'(\pi'(\ell')) = \tau'(K+\ell')$$

and thus,

$$\tau^{-1}\circ\tau'(K+\ell') = K + \pi^{-1}\circ\pi'(\ell').$$

We conclude that

$$\begin{aligned} \operatorname{sgn}(\tau)\operatorname{sgn}(\tau') &= \operatorname{sgn}(\tau^{-1}\circ\tau') \\ &= \operatorname{sgn}(\sigma^{-1}\circ\sigma')\operatorname{sgn}(\pi^{-1}\circ\pi') = \operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)\operatorname{sgn}(\sigma')\operatorname{sgn}(\pi'), \end{aligned}$$

from which it follows that

$$\left(\frac{\operatorname{sgn}(\tau)}{\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)}\right)\left(\frac{\operatorname{sgn}(\tau')}{\operatorname{sgn}(\sigma')\operatorname{sgn}(\pi')}\right) = 1.$$

Since τ' was arbitrary we must conclude that the definition of $\operatorname{sgn}(\mathfrak{t})$ is independent of the permutation $\tau \in S_N$ used to define it.

Given $\mathfrak{t} \in P_K^N$ there is a distinguished permutation $\iota_\mathfrak{t} \in S_K(\mathfrak{t})$ specified by

$$\iota_\mathfrak{t}(n) = \begin{cases} \mathfrak{t}(n) & \text{if} \quad 1 \le n \le K \\ \mathfrak{t}'(n-K) & \text{if} \quad K < n \le N. \end{cases}$$

This permutation induces the identity permutation on both S_K and S_{N-K}, and hence $\operatorname{sgn}(\mathfrak{t}) = \operatorname{sgn}(\iota_\mathfrak{t})$.

We also introduce the distinguished element $\mathfrak{i} \in P_K^N$ given by $\mathfrak{i}(k) = k$ for $1 \le k \le K$. It is easily verified that $\operatorname{sgn}(\mathfrak{i}) = 1$.

The following lemma will be convenient for us. The proof of this lemma demonstrates the usefulness of the fact that $\operatorname{sgn}(\mathfrak{t}) = \operatorname{sgn}(\iota_\mathfrak{t})$.

Lemma 5.11. *Suppose $J \geq 1$. Then,*

$$\sum_{\mathfrak{t} \in P_2^{2J}} \operatorname{sgn}(\mathfrak{t}) = J.$$

It should be remarked that this is a special case of the more general fact which asserts that if $0 < K < J$, then

$$\sum_{\mathfrak{t} \in P_{2K}^{2J}} \operatorname{sgn}(\mathfrak{t}) = \binom{J}{K}.$$

This fact will be proved later, and its proof requires us first to know the $K = 1$ case.

Proof. Let $\mathfrak{t} \in P_2^{2J}$ with $\mathfrak{t}(1) = n_1, \mathfrak{t}(2) = n_2$. We will first prove that

$$\operatorname{sgn}(\mathfrak{t}) = \begin{cases} 1 & \text{if } (n_2 - n_1) \equiv 1 \bmod 2 \\ -1 & \text{if } (n_2 - n_1) \equiv 0 \bmod 2 \end{cases} \tag{5.14}$$

There are J^2 ways of choosing two integers from $\{1, 2, \ldots, 2J\}$ such that one of the integers is even and the other is odd. There are $J(J-1)$ ways of choosing two integers from $\{1, 2, \ldots, 2J\}$ such that both integers are even or both integers are odd. This together with Equation 5.14 establishes the lemma.

Equation 5.14 will be verified in cases. First assume that $2 < n_1$. We remark that $n_1 < n_2$ by the definition of $\mathfrak{t}$. We construct $\iota_{\mathfrak{t}}$:

$$\iota_{\mathfrak{t}}(n) = \begin{cases} n_1 & \text{if } \quad n = 1 \\ n_2 & \text{if } \quad n = 2 \\ n-2 & \text{if } \quad 3 \leq n \leq n_1 + 1 \\ n-1 & \text{if } \quad n_1 + 2 \leq n \leq n_2 \\ n & \text{if } \quad n_2 + 1 \leq n \leq 2J. \end{cases}$$

If n_1 is odd then the cycle decomposition of $\iota_{\mathfrak{t}}$ is given by

$$\begin{aligned} &(n_1 \quad n_1{-}2 \quad n_1{-}4 \quad \cdots \quad 3 \quad 1) \\ &\quad \cdot (n_2 \quad n_2{-}1 \quad n_2{-}2 \quad \cdots \quad n_1{+}1 \quad n_1{-}1 \quad n_1{-}3 \quad \cdots \quad 4 \quad 2). \end{aligned}$$

The first cycle decomposes into $(n_1-1)/2$ transpositions, and the second cycle decomposes into $(n_1-1)/2+(n_2-n_1-1)$ transpositions. Thus if n_1 and n_2 are both odd then $\operatorname{sgn}(\mathfrak{t})=-1$, and if n_1 is odd and n_2 is even then $\operatorname{sgn}(\mathfrak{t})=1$.

If n_1 is even the cycle decomposition of $\iota_{\mathfrak{t}}$ is given by

$$(n_1 \quad n_1{-}2 \quad \cdots \quad 4 \quad 2 \quad n_2 \quad n_2{-}1 \quad \cdots \quad n_1{+}1 \quad n_1{-}1 \quad n_1{-}3 \quad \cdots \quad 3 \quad 1).$$

In this case $\iota_{\mathfrak{t}}$ decomposes into n_2-1 transpositions. It follows that if n_1 and n_2 are both even then $\operatorname{sgn}(\mathfrak{t})=-1$, and if n_1 is even and n_2 is odd then $\operatorname{sgn}(\mathfrak{t})=1$.

In the case where $\mathfrak{t}(1)=1$, it can be demonstrated that the cycle decomposition of $\iota_{\mathfrak{t}}$ is given by

$$(n_2 \quad n_2-1 \quad n_2-2 \quad \cdots \quad 3 \quad 2),$$

which decomposes into n_2-2 transpositions. Thus in this case $\operatorname{sgn}(\mathfrak{t})=1$ if n_2 is even and $\operatorname{sgn}(\mathfrak{t})=-1$ if n_2 is odd.

In the case where $\mathfrak{t}(1)=2$, it can be demonstrated that the cycle decomposition of $\iota_{\mathfrak{t}}$ is given by

$$(n_2 \quad n_2-1 \quad n_2-2 \quad \cdots \quad 4 \quad 3 \quad 1 \quad 2),$$

which decomposes into n_2-1 transpositions. Thus in this case $\operatorname{sgn}(\mathfrak{t})=-1$ if n_2 is even and $\operatorname{sgn}(\mathfrak{t})=1$ if n_2 is odd.

Obviously, if $\mathfrak{t}(1)=1$ and $\mathfrak{t}(2)=2$, then $\iota_{\mathfrak{t}}$ is the identity permutation, and $\operatorname{sgn}(\mathfrak{t})=1$. $\square$

5.4.3 Determinants and Pfaffians

Given an $N\times N$ matrix W and $\mathfrak{s},\mathfrak{t}\in P_K^N$, define $W_{\mathfrak{s},\mathfrak{t}}$ to be the $K\times K$ minor whose j,k entry is given by $W_{\mathfrak{s},\mathfrak{t}}(j,k)=W(\mathfrak{s}(j),\mathfrak{t}(k))$. It is easily seen that the complimentary $(N-K)\times(N-K)$ minor is given by $W_{\mathfrak{s}',\mathfrak{t}'}$.

Lemma 5.12 (Laplace expansion of the determinant)**.** *Let W be an $N \times N$ matrix, and let $K < N$ be a positive integer. Then, for any $\mathfrak{s} \in P_K^N$,*

$$\det(W) = \operatorname{sgn}(\mathfrak{s}) \sum_{\mathfrak{t} \in P_K^N} \operatorname{sgn}(\mathfrak{t}) \det(W_{\mathfrak{s},\mathfrak{t}}) \det(W_{\mathfrak{s}',\mathfrak{t}'}).$$

Proof. We use the well–known formula for the determinant:

$$\det(W) = \sum_{\tau \in S_N} \operatorname{sgn}(\tau) \prod_{n=1}^{N} W(n, \tau(n)) = \operatorname{sgn}(\iota_{\mathfrak{s}}) \sum_{\tau \in S_N} \operatorname{sgn}(\tau) \prod_{n=1}^{N} W(\iota_{\mathfrak{s}}(n), \tau(n)).$$

Since S_N is partitioned by $\{S_N(\mathfrak{t}) : \mathfrak{t} \in P(K)\}$ we may write

$$\begin{aligned}
\det(W) &= \operatorname{sgn}(\mathfrak{s}) \sum_{\mathfrak{t} \in P_K^N} \sum_{\tau \in S_N(\mathfrak{t})} \operatorname{sgn}(\tau) \prod_{j=1}^{K} W(\mathfrak{s}(j), \tau(j)) \prod_{\ell=1}^{N-K} W(\mathfrak{s}'(\ell), \tau(K+\ell)) \\
&= \operatorname{sgn}(\mathfrak{s}) \sum_{\mathfrak{t} \in P_K^N} \sum_{\tau \in S_N(\mathfrak{t})} \operatorname{sgn}(\tau) \prod_{j=1}^{K} W(\mathfrak{s}(j), \mathfrak{t} \circ \sigma_\tau(j)) \times \prod_{\ell=1}^{N-K} W(\mathfrak{s}'(\ell), \mathfrak{t}' \circ \pi_\tau(\ell)),
\end{aligned}$$

where the last equality follows from the definition of σ_τ and π_τ.

As τ ranges over all of $S_N(\mathfrak{t})$, σ_τ and π_τ range over all of S_K and S_{N-K} (resp.). This together with the definition of $\operatorname{sgn}(\mathfrak{t})$ allows us to remove the dependence on τ to find that

$$\begin{aligned}
\det(W) = \operatorname{sgn}(\mathfrak{s}) \sum_{\mathfrak{t} \in P_K^N} \operatorname{sgn}(\mathfrak{t}) &\left\{ \sum_{\sigma \in S_K} \operatorname{sgn}(\sigma) \prod_{j=1}^{K} W(\mathfrak{s}(j), \mathfrak{t} \circ \sigma(j)) \right\} \\
&\times \left\{ \sum_{\pi \in S_{N-K}} \operatorname{sgn}(\pi) \prod_{\ell=1}^{N-K} W(\mathfrak{s}'(\ell), \mathfrak{t}' \circ \pi(\ell)) \right\}.
\end{aligned}$$

The terms in brackets are exactly $\det(W_{\mathfrak{s},\mathfrak{t}})$ and $\det(W_{\mathfrak{s}',\mathfrak{t}'})$ (resp.). □

We next prove three lemmas about Pfaffians. Let $K \leq J$ be positive integers and let U be a $2J \times 2J$ anti-symmetric matrix, and let $\mathfrak{t} \in P_{2K}^{2J}$. The minor $U_{\mathfrak{t},\mathfrak{t}}$ is an antisymmetric $2K \times 2K$ matrix, and hence we may talk about its Pfaffian. We will abbreviate our notation by specifying that $U_{\mathfrak{t}} = U_{\mathfrak{t},\mathfrak{t}}$. The first lemma about Pfaffians is an analog of the Laplace expansion of the determinant.

Lemma 5.13 (Laplace expansion of the Pfaffian)**.** *Let U be a $2J \times 2J$ anti-symmetric matrix, and let $K < J$ be a positive integer. Then,*

$$\operatorname{Pf}(U) = \binom{J}{K}^{-1} \sum_{\mathfrak{t} \in P_{2K}^{2J}} \operatorname{sgn}(\mathfrak{t}) \operatorname{Pf}(U_{\mathfrak{t}}) \operatorname{Pf}(U_{\mathfrak{t}'}).$$

Proof. As in the proof of Lemma 5.12 we use the fact that S_{2J} is partitioned by $\{S_{2J}(\mathfrak{t}) : \mathfrak{t} \in P_{2K}^{2J}\}$ to write

$$\begin{aligned} \operatorname{Pf}(U) &= \frac{1}{2^J J!} \sum_{\mathfrak{t} \in P_{2K}^{2J}} \sum_{\tau \in S_{2J}(\mathfrak{t})} \operatorname{sgn}(\tau) \left\{ \prod_{n=1}^{K} U(\tau(2n-1), \tau(2n)) \right\} \\ &\qquad \times \left\{ \prod_{\ell=1}^{J-K} U(\tau(2K+2\ell-1), \tau(2K+2\ell)) \right\}. \end{aligned}$$

Using the definition of σ_τ, π_τ and $\operatorname{sgn}(\mathfrak{t})$ we find

$$\begin{aligned} \operatorname{Pf}(U) &= \frac{1}{J!} \sum_{\mathfrak{t} \in P_{2K}^{2J}} \operatorname{sgn}(\mathfrak{t}) \left\{ \frac{1}{2^K} \sum_{\sigma \in S_{2K}} \operatorname{sgn}(\sigma) \prod_{n=1}^{K} U(\mathfrak{t}\circ\sigma(2n-1), \mathfrak{t}\circ\sigma(2n)) \right\} \\ &\qquad \times \left\{ \frac{1}{2^{J-K}} \sum_{\pi \in S_{2(J-K)}} \operatorname{sgn}(\pi) \prod_{\ell=1}^{J-K} U(\mathfrak{t}'\circ\pi(2\ell-1), \mathfrak{t}'\circ\pi(2\ell)) \right\}. \end{aligned}$$

After multiplying the terms in braces by $1/K!$ and $1/(J-K)!$ (resp.), we find the expressions for $\operatorname{Pf}(U_{\mathfrak{t}})$ and $\operatorname{Pf}(U_{\mathfrak{t}'})$ (resp.). Altering the constant in front of the outermost sum to compensate, we arrive at the formulation for $\operatorname{Pf}(U)$ given in the statement of the lemma. ∎

Lemma 5.14. *Let R and C be two anti-symmetric $2J \times 2J$ matrices. Then,*

$$\operatorname{Pf}(R + C) = \sum_{K=0}^{J} \sum_{\mathfrak{t} \in P_{2K}^{2J}} \operatorname{sgn}(\mathfrak{t}) \operatorname{Pf}(R_{\mathfrak{t}}) \operatorname{Pf}(C_{\mathfrak{t}'}).$$

Proof. Let $U = R + C$. Then,

$$\mathrm{Pf}(U) = \frac{1}{2^J J!} \sum_{\tau \in S_{2J}} \mathrm{sgn}(\tau) \left\{ \prod_{j=1}^{J} R(\tau(2n-1), \tau(2n)) + C(\tau(2n-1), \tau(2n)) \right\}.$$

We may expand the product in this expression, by noting that,

$$\begin{aligned}
&\prod_{j=1}^{J} R(\tau(2n-1), \tau(2n)) + C(\tau(2n-1), \tau(2n)) \\
&\quad = \sum_{K=0}^{J} \left\{ \sum_{\mathfrak{t} \in P_K^J} \prod_{k=1}^{K} R(\tau(2\mathfrak{t}(k)-1), \tau(2\mathfrak{t}(k))) \prod_{\ell=1}^{J-K} C(\tau(2\mathfrak{t}'(\ell)-1), \tau(2\mathfrak{t}'(\ell))) \right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{Pf}(U) = \frac{1}{2^J J!} \sum_{K=0}^{J} \sum_{\mathfrak{t} \in P_K^J} \sum_{\tau \in S_{2J}} \mathrm{sgn}(\tau) &\prod_{k=1}^{K} R(\tau(2\mathfrak{t}(k)-1), \tau(2\mathfrak{t}(k))) \\
\times &\prod_{\ell=1}^{J-K} C(\tau(2\mathfrak{t}'(\ell)-1), \tau(2\mathfrak{t}'(\ell)))
\end{aligned}$$

Next, we use $\mathfrak{t}$ to determine a permutation $\rho_{\mathfrak{t}}$ in S_{2J}, where

$$\rho_{\mathfrak{t}}(2k-1) = 2\mathfrak{t}(k) - 1, \quad \rho_{\mathfrak{t}}(2k) = 2\mathfrak{t}(k) \qquad k = 1, 2, \ldots, K,$$

and

$$\rho_{\mathfrak{t}}(2K + 2\ell - 1) = 2\mathfrak{t}'(\ell) - 1, \quad \rho_{\mathfrak{t}}(2K + 2\ell) = 2\mathfrak{t}'(\ell) \qquad \ell = 1, 2, \ldots, J - K.$$

It is easily verified that $\mathrm{sgn}(\rho_{\mathfrak{t}}) = \mathrm{sgn}(\mathfrak{t})^2$, and hence

$$\begin{aligned}
\mathrm{Pf}(U) \quad = \quad & \frac{1}{2^J J!} \sum_{K=0}^{J} \sum_{\mathfrak{t} \in P_K^J} \sum_{\tau \in S_{2J}} \mathrm{sgn}(\tau \circ \rho_{\mathfrak{t}}) \prod_{k=1}^{K} R(\tau \circ \rho_{\mathfrak{t}}(2k-1), \tau \circ \rho_{\mathfrak{t}}(2k)) \\
& \times \prod_{\ell=1}^{J-K} C(\tau \circ \rho_{\mathfrak{t}}(2K + 2m - 1), \tau \circ \rho_{\mathfrak{t}}(2K + 2\ell)).
\end{aligned}$$

Now, we recognize that the inner sum is independent of $\mathfrak{t}$ since it may be reindexed to remove the dependence on $\rho_{\mathfrak{t}}$. Thus, since the cardinality of P_K^J is $\binom{J}{K}$ we find,

$$\begin{aligned}\mathrm{Pf}(U) &= \frac{1}{2^J}\sum_{K=0}^{J}\frac{1}{K!(J-K)!}\sum_{\tau\in S_{2J}}\mathrm{sgn}(\tau)\prod_{k=1}^{K}R(\tau(2k-1),\tau(2k)) \\ &\quad\times\prod_{\ell=1}^{J-K}C(\tau(2K+2\ell-1),\tau(2K+2\ell)).\end{aligned}$$

Now, for each $K\geq 0$, $\{S_{2J}(\mathfrak{t}):\mathfrak{t}\in P_{2K}^{2J}\}$ partitions S_{2J}. Thus

$$\begin{aligned}\mathrm{Pf}(A) &= \frac{1}{2^J}\sum_{K=0}^{J}\sum_{\mathfrak{t}\in P_{2K}^{2J}}\frac{1}{K!(J-K)!}\sum_{\tau\in S_{2J}(\mathfrak{t})}\mathrm{sgn}(\tau)\prod_{k=1}^{K}R(\tau(2k-1),\tau(2k)) \\ &\quad\times\prod_{\ell=1}^{J-K}C(\tau(2K+2\ell-1),\tau(2K+2\ell)).\end{aligned}$$

As in Lemma 5.12 we may replace the sum over $S_{2J}(\mathfrak{t})$ with sums over S_{2K} and $S_{2(J-K)}$ to find,

$$\begin{aligned}\mathrm{Pf}(U) = \sum_{K=0}^{J}\sum_{\mathfrak{t}\in P_{2K}^{2J}}\mathrm{sgn}(\mathfrak{t})&\left\{\frac{1}{2^K K!}\sum_{\sigma\in S_{2K}}\mathrm{sgn}(\sigma)\prod_{k=1}^{K}R\left(\mathfrak{t}\circ\sigma(2k+1),\mathfrak{t}\circ\sigma(2k)\right)\right\} \\ \times&\left\{\frac{1}{2^{J-K}(J-K)!}\sum_{\pi\in S_{2(J-K)}}\mathrm{sgn}(\pi)\prod_{\ell=1}^{J-K}C\left(\mathfrak{t}'\circ\pi(2\ell-1),\mathfrak{t}'\circ\pi(2\ell)\right)\right\}.\end{aligned}$$

Both terms in braces can be written as Pfaffians in their own right, and after comparing this expression with those for $\mathrm{Pf}(R_{\mathfrak{t}}^{\mathfrak{t}})$ and $\mathrm{Pf}(C_{\mathfrak{t}'}^{\mathfrak{t}'})$ we find the expression for $\mathrm{Pf}(U)$ given in the statement of the lemma. □

The next lemma allows us to write the Pfaffian of certain matrices as a determinant.

Lemma 5.15. *Suppose $N=2J$, and U is an $N\times N$ anti-symmetric matrix such that $U(j,k)=0$ if $(j-k)\equiv 0 \bmod 2$. Then,*

$$\mathrm{Pf}(U)=\det(A)$$

where A is the $J \times J$ matrix whose j, k entry is given by $A(j,k) = U(2j-1, 2k)$.

Proof. From the definition of Pfaffian,

$$\mathrm{Pf}(U) = \frac{1}{2^J J!} \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{J} U(\sigma(2j-1), \sigma(2j)). \tag{5.15}$$

Clearly the product in this expression is 0 except for permutations σ such that $\sigma(2j-1) - \sigma(2j) \equiv 1 \bmod 2$. Let G_N denote the subgroup of S_N given by

$$G_N = \{\sigma \in S_N : (\sigma(n) - n) \equiv 0 \bmod 2, \quad n = 1, 2, \ldots, N\}.$$

Given $\sigma \in S_N$ such that $\sigma(2j-1) - \sigma(2j) \equiv 1 \bmod 2$, define $\sigma^* \in G_N$ by

$$\sigma^*(2j) = \begin{cases} \sigma(2j) & \text{if } \sigma(2j) \text{ is even} \\ \sigma(2j-1) & \text{if } \sigma(2j) \text{ is odd,} \end{cases}$$

and

$$\sigma^*(2j-1) = \begin{cases} \sigma(2j) & \text{if } \sigma(2j) \text{ is odd} \\ \sigma(2j-1) & \text{if } \sigma(2j) \text{ is even.} \end{cases}$$

Notice that σ and σ^* differ only by a product of transpositions of the form $(2j-1, 2j)$ where $j = 1, 2, \ldots, J$. Moreover, since U is an antisymmetric matrix,

$$\mathrm{sgn}(\sigma) \prod_{j=1}^{J} U(\sigma(2j-1), \sigma(2j)) = \mathrm{sgn}(\sigma^*) \prod_{j=1}^{J} U(\sigma^*(2j-1), \sigma^*(2j)).$$

Clearly, each $\sigma^* \in G_N$ represents 2^J different permutations $\sigma \in S_N$ which satisfy $\sigma(2j-1) - \sigma(2j) \equiv 1 \bmod 2$. We may thus replace the sum over S_N in (5.15) with a sum over G_N to find

$$\mathrm{Pf}(U) = \frac{1}{J!} \sum_{\sigma^* \in G_N} \mathrm{sgn}(\sigma^*) \prod_{j=1}^{J} U(\sigma^*(2j-1), \sigma^*(2j)).$$

Now, since elements of G_N permute even integers and odd integers disjointly we have G_N is isomorphic to $S_J \times S_J$, and hence

$$\mathrm{Pf}(U) = \frac{1}{J!} \sum_{\tau \in S_J} \sum_{\pi \in S_J} \mathrm{sgn}(\tau)\, \mathrm{sgn}(\pi) \prod_{j=1}^{J} U(2\tau(j)-1, 2\pi(j)).$$

But, by [15, Lemma 3.1] this is exactly $\det A$. $\square$

5.4.4 The Absolute Value of the Vandermonde

Given a complete family of monic polynomials $Q \subseteq \mathbb{R}[\gamma]$ define the $N \times N$ matrix $W^{\boldsymbol{\gamma}}$ by specifying that the j, k entry of $W^{\boldsymbol{\gamma}}$ is given by $W^{\boldsymbol{\gamma}}(j,k) = Q_k(\gamma_j)$. Of course $W^{\boldsymbol{\gamma}}$ also depends on Q, but this dependence will not be emphasized at this point since we are primarily interested in the fact that $V(\boldsymbol{\gamma}) = \det(W^{\boldsymbol{\gamma}})$.

Lemma 5.16. *Suppose $L + 2M = N$, and let $\boldsymbol{\alpha} \in \mathbb{R}^L$, $\boldsymbol{\beta} \in \mathbb{C}^M$. Set*

$$\boldsymbol{\gamma} = (\alpha_1, \ldots, \alpha_L, \overline{\beta_1}\beta_1, \ldots, \overline{\beta_M}, \beta_M),$$

and define $V(\boldsymbol{\alpha}, \boldsymbol{\beta}) = V(\boldsymbol{\gamma})$. Then $|V(\boldsymbol{\alpha}, \boldsymbol{\beta})|$ can be given by

$$\sum_{\mathfrak{t} \in P_L^N} \operatorname{sgn}(\mathfrak{t}) \left\{ \det(A^{\boldsymbol{\alpha}}_{\mathfrak{i},\mathfrak{t}}) \prod_{1 \le j < k \le L} \operatorname{sgn}(\alpha_k - \alpha_j) \right\} \times \left\{ \det(B^{\boldsymbol{\beta}}_{\mathfrak{i}',\mathfrak{t}'})(-i)^M \prod_{m=1}^{M} \operatorname{sgn}(\Im(\beta_m)) \right\},$$

where $A^{\boldsymbol{\alpha}}_{\mathfrak{i},\mathfrak{t}}$ is the $L \times L$ minor $W^{\boldsymbol{\gamma}}_{\mathfrak{i},\mathfrak{t}}$, and $B^{\boldsymbol{\beta}}_{\mathfrak{i}',\mathfrak{t}'}$ is the $2M \times 2M$ minor $W^{\boldsymbol{\gamma}}_{\mathfrak{i}',\mathfrak{t}'}$.

As suggested by the notation, for each $\mathfrak{t} \in P_L^N$, the entries of $A^{\boldsymbol{\alpha}}_{\mathfrak{i},\mathfrak{t}}$ are dependent only on $\boldsymbol{\alpha}$, and the entries of $B^{\boldsymbol{\beta}}_{\mathfrak{i}',\mathfrak{t}'}$ are dependent only on $\boldsymbol{\beta}$.

Proof. By (5.11) we see that

$$V(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left\{ \prod_{j<k} (\alpha_k - \alpha_j) \right\} \prod_{l=1}^{L} \prod_{m=1}^{M} |\beta_m - \alpha_l|^2 \times \left\{ \prod_{m<n} |\beta_n - \beta_m|^2 \left|\beta_n - \overline{\beta_m}\right|^2 \right\} \prod_{m=1}^{M} (2i\Im(\beta_m)).$$

And hence,

$$|V(\boldsymbol{\alpha}, \boldsymbol{\beta})| = (-i)^M \left\{ \prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \operatorname{sgn}(\Im(\beta_m)) \right\} V(\boldsymbol{\alpha}, \boldsymbol{\beta}). \qquad (5.16)$$

Now, using the Laplace expansion of the determinant of $W^{\boldsymbol{\gamma}}$ we see that

$$V(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\mathfrak{t} \in P_L^N} \operatorname{sgn}(\mathfrak{t}) \det(W^{\boldsymbol{\gamma}}_{\mathfrak{i},\mathfrak{t}}) \det(W^{\boldsymbol{\gamma}}_{\mathfrak{i}',\mathfrak{t}'}).$$

Combining this with (5.16) we arrive at the formulation of $|V(\boldsymbol{\alpha}, \boldsymbol{\beta})|$ given in the statement of the lemma. □

Lemma 5.17. *Suppose L is even and let $\boldsymbol{\alpha} \in \mathbb{R}^L$. Let $T^{\boldsymbol{\alpha}}$ be the $L \times L$ matrix whose j, k entry is given by $T^{\boldsymbol{\alpha}}(j, k) = \operatorname{sgn}(\alpha_k - \alpha_j)$. Then,*

$$\prod_{1 \le j < k \le L} \operatorname{sgn}(\alpha_k - \alpha_j) = \operatorname{Pf}(T^{\boldsymbol{\alpha}}).$$

Proof. It is easily seen that the lemma is true in the case where $\alpha_j = \alpha_k$ for some $j \neq k$. Thus we will assume that the α_j are distinct we will first prove this lemma in the case $\alpha_1 < \alpha_2 < \ldots < \alpha_L$. In this case $T^{\boldsymbol{\alpha}}$ is the $L \times L$ matrix given by

$$T^{\boldsymbol{\alpha}} = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & 0 \end{pmatrix}. \tag{5.17}$$

In this case we are trying to prove that the Pfaffian of the right hand side of (5.17) is 1. Let $L = 2J$. We will induct on J. The inductive hypothesis states that if $M < J$, then the Pfaffian of the $2M \times 2M$ matrix formed as in the right hand side of (5.17) is equal to 1. The base case of the induction is easy, since it is straightforward to verify that

$$\operatorname{Pf}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1. \tag{5.18}$$

We now use Lemma 5.13 with $K = 1$ to show

$$\operatorname{Pf}(T^{\boldsymbol{\alpha}}) = \frac{1}{J} \sum_{\mathfrak{t} \in P_2^{2J}} \operatorname{sgn}(\mathfrak{t}) \operatorname{Pf}(T^{\boldsymbol{\alpha}}_{\mathfrak{t}}) \operatorname{Pf}(T^{\boldsymbol{\alpha}}_{\mathfrak{t}'}).$$

But each minor of the form $T^{\boldsymbol{\alpha}}_{\mathfrak{t}}$ is a 2×2 anti-symmetric matrix identical to the matrix in the left hand side of Equation 5.18, and each matrix of the form $T^{\boldsymbol{\alpha}}_{\mathfrak{t}}$ is a $2(J-1) \times 2(J-1)$ matrix of the form given in the right hand side of Equation 5.17. Thus, by the inductive hypothesis we have

$$\mathrm{Pf}(T^{\boldsymbol{\alpha}}) = \frac{1}{J} \sum_{\mathfrak{t} \in P_2^{2J}} \mathrm{sgn}(\mathfrak{t}).$$

But then, Lemma 5.11 gives us that $\mathrm{Pf}(T^{\boldsymbol{\alpha}}) = 1$.

To complete the proof we use the action of S_L on $\mathbb{R}^L$ which is given by $\sigma \cdot \boldsymbol{\alpha} = (\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(L)})$. And, for any $\boldsymbol{\alpha} \in \mathbb{R}^L$ with distinct coordinates there is a $\sigma \in S_L$ such that $T^{\sigma \cdot \boldsymbol{\alpha}}$ is given as in the right hand side of (5.17). But then, it is easily verified, by writing out the definition of $\mathrm{Pf}(T^{\sigma \cdot \boldsymbol{\alpha}})$ that

$$\mathrm{Pf}(T^{\boldsymbol{\alpha}}) = \mathrm{sgn}(\sigma)\,\mathrm{Pf}(T^{\sigma \cdot \boldsymbol{\alpha}}) = \mathrm{sgn}(\sigma) = \prod_{1 \le j < k \le L} \mathrm{sgn}(\alpha_k - \alpha_j).$$

□

Corollary 5.18. *Suppose $K < J$ are positive integers, and $N = 2J$. Then,*

$$\sum_{\mathfrak{t} \in P_{2K}^{2J}} \mathrm{sgn}(\mathfrak{t}) = \binom{J}{K}.$$

Proof. Let $\boldsymbol{\alpha} \in \mathbb{R}^L$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_L$. We now know that $\mathrm{Pf}(T^{\boldsymbol{\alpha}}) = 1$. Thus, using the Laplace expansion of the Pfaffian with $K < J$, we have

$$1 = \mathrm{Pf}(T^{\boldsymbol{\alpha}}) = \binom{J}{K}^{-1} \sum_{\mathfrak{t} \in P_{2K}^{2J}} \mathrm{sgn}(\mathfrak{t})\,\mathrm{Pf}(T^{\boldsymbol{\alpha}}_{\mathfrak{t}})\,\mathrm{Pf}(T^{\boldsymbol{\alpha}}_{\mathfrak{t}'}) = \binom{J}{K}^{-1} \sum_{\mathfrak{t} \in P_{2K}^{2J}} \mathrm{sgn}(\mathfrak{t}).$$

□

Corollary 5.19. *Suppose $L + 2M = N$, and let $\boldsymbol{\alpha} \in \mathbb{R}^L$, $\boldsymbol{\beta} \in \mathbb{C}^M$. Then,*

$$|V(\boldsymbol{\alpha}, \boldsymbol{\beta})| = \det(B^{\boldsymbol{\beta}}_{\mathfrak{i}',\mathfrak{t}'})(-i)^M \prod_{m=1}^{M} \mathrm{sgn}(\Im(\beta_m)) \sum_{\mathfrak{t} \in P_L^N} \mathrm{sgn}(\mathfrak{t}) \left\{ \det(A^{\boldsymbol{\alpha}}_{\mathfrak{i},\mathfrak{t}})\,\mathrm{Pf}(T^{\boldsymbol{\alpha}}) \right\},$$

where $A^{\boldsymbol{\alpha}}_{\mathfrak{i},\mathfrak{t}}$ and $B^{\boldsymbol{\beta}}_{\mathfrak{i}',\mathfrak{t}'}$ are given as in Lemma 5.16.

5.4.5 Integrals as Pfaffians

As before, let N be even with $L + 2M = N$ and let $R = R_Q$ and $C = C_Q$ be the $N \times N$ matrices given as in Equation 5.1. We will use Fubini's Theorem to write the Pfaffians of $L \times L$ minors of R_Q and the $2M \times 2M$ minors of C_Q as multiple integrals over $\mathbb{R}^L$ and $\mathbb{C}^M$ (resp.).

Lemma 5.20. *Let $\mathfrak{t} \in P_L^N$, and let $B^{\boldsymbol{\beta}}_{\mathfrak{i}',\mathfrak{t}'}$ be as in Lemma 5.16. Then,*

$$\mathrm{Pf}(C_{\mathfrak{t}'}) = \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det(B^{\boldsymbol{\beta}}_{\mathfrak{i}',\mathfrak{t}'}) \prod_{m=1}^{M} \phi(\overline{\beta_m})^{-s} \phi(\beta_m)^{-s} \operatorname{sgn}(\Im(\beta_m))\, d\lambda_{2M}(\boldsymbol{\beta}). \tag{5.19}$$

Proof. From the definition of $B^{\boldsymbol{\beta}}_{\mathfrak{i}',\mathfrak{t}'}$ we can write

$$\det(B^{\boldsymbol{\beta}}_{\mathfrak{i}'\mathfrak{t}'}) = \sum_{\pi \in S_{2M}} \operatorname{sgn}(\pi) \prod_{m=1}^{M} Q_{\mathfrak{t}' \circ \pi(2m-1)}(\overline{\beta_m})\; Q_{\mathfrak{t}' \circ \pi(2m)}(\beta_m).$$

And thus the right hand side of (5.19) as

$$\frac{1}{M!} \sum_{\pi \in S_{2M}} \operatorname{sgn}(\pi)(-i)^M \int_{\mathbb{C}^M} \left\{ \prod_{m=1}^{M} \phi(\overline{\beta_m})^{-s} \phi(\beta_m)^{-s} \operatorname{sgn}(\Im(\beta_m)) \right\}$$
$$\times \left\{ \prod_{m=1}^{M} Q_{\mathfrak{t}' \circ \pi(2m-1)}(\overline{\beta_m})\; Q_{\mathfrak{t}' \circ \pi(2m)}(\beta_m) \right\} d\lambda_{2M}(\boldsymbol{\beta}).$$

When $\Re(s) > N$, this integral converges, and hence we may use Fubini's Theorem to write this expression as

$$\frac{1}{2^M M!} \sum_{\pi \in S_{2M}} \operatorname{sgn}(\pi) \prod_{m=1}^{M} -2i \int_{\mathbb{C}} \phi(\overline{\beta})^{-s} \phi(\beta)^{-s}$$
$$\times Q_{\mathfrak{t}' \circ \pi(2m-1)}(\overline{\beta})\; Q_{\mathfrak{t}' \circ \pi(2m)}(\beta) \operatorname{sgn}(\Im(\beta))\, d\lambda_2(\beta)$$

which equals

$$\frac{1}{2^M M!} \sum_{\pi \in S_{2M}} \operatorname{sgn}(\pi) \prod_{m=1}^{M} \langle Q_{\mathfrak{t}' \circ \pi(2m-1)}, Q_{\mathfrak{t}' \circ \pi(2m)} \rangle_{\mathbb{C}},$$

which is exactly the definition of $\mathrm{Pf}(C_{\mathfrak{t}'})$. $\square$

Lemma 5.21. *Suppose L is even. Let $\mathfrak{t} \in P_L^N$, and let $A_{\mathfrak{i},\mathfrak{t}}^{\boldsymbol{\alpha}}$ be as in Lemma 5.16, and let $T^{\boldsymbol{\alpha}}$ be as in Lemma 5.17. Then,*

$$\mathrm{Pf}(R_{\mathfrak{t}}) = \frac{1}{L!} \int_{\mathbb{R}^L} \det(A_{\mathfrak{i},\mathfrak{t}}^{\boldsymbol{\alpha}}) \, \mathrm{Pf}(T^{\boldsymbol{\alpha}}) \prod_{\ell=1}^{L} \phi(\alpha_\ell)^{-s} \, d\lambda_L(\boldsymbol{\alpha}). \tag{5.20}$$

Proof. We expand $\det(A_{\mathfrak{i},\mathfrak{t}}^{\boldsymbol{\alpha}})$ as a sum over S_L to write the right hand side of (5.20) as

$$\frac{1}{L!} \sum_{\sigma \in S_L} \mathrm{sgn}(\sigma) \left\{ \int_{\mathbb{R}^L} \prod_{\ell=1}^{L} \phi(\alpha_\ell)^{-s} \prod_{\ell=1}^{L} Q_{\mathfrak{t}(\ell)}(\alpha_{\sigma(\ell)}) \; \mathrm{Pf}(T^{\boldsymbol{\alpha}}) \, d\lambda_L(\boldsymbol{\alpha}) \right\}. \tag{5.21}$$

Recalling that for each $\sigma \in S_L$, $\mathrm{Pf}(T^{\sigma \cdot \boldsymbol{\alpha}}) = \mathrm{sgn}(\sigma) \, \mathrm{Pf}(T^{\boldsymbol{\alpha}})$, we use the change of variables $\alpha \mapsto \sigma^{-1} \cdot \alpha$ to write the integral in braces in (5.21) as

$$\mathrm{sgn}(\sigma^{-1}) \int_{\mathbb{R}^L} \prod_{\ell=1}^{L} \phi(\alpha_\ell)^{-s} \prod_{\ell=1}^{L} Q_{\mathfrak{t}(\ell)}(\alpha_\ell) \; \mathrm{Pf}(T^{\boldsymbol{\alpha}}) \, d\lambda_L(\boldsymbol{\alpha}).$$

Substituting this into (5.21) we may remove the sum over S_L to write (5.21) as

$$\int_{\mathbb{R}^L} \prod_{\ell=1}^{L} \phi(\alpha_\ell)^{-s} \prod_{\ell=1}^{L} Q_{\mathfrak{t}(\ell)}(\alpha_\ell) \; \mathrm{Pf}(T^{\boldsymbol{\alpha}}) \, d\lambda_L(\boldsymbol{\alpha}).$$

We set $K = L/2$, and expand $\mathrm{Pf}(T^{\boldsymbol{\alpha}})$ to write this as

$$\frac{1}{2^K K!} \sum_{\tau \in S_{2K}} \mathrm{sgn}(\tau) \int_{\mathbb{R}^{2K}} \prod_{\ell=1}^{2K} \phi(\alpha_\ell)^{-s} Q_{\mathfrak{t}(\ell)}(\alpha_\ell) \prod_{k=1}^{K} \mathrm{sgn}(\alpha_{\tau(2k)} - \alpha_{\tau(2k-1)}) \, d\lambda_L(\boldsymbol{\alpha}). \tag{5.22}$$

But of course, for each $\tau \in S_{2K}$,

$$\prod_{\ell=1}^{2K} \phi(\alpha_\ell)^{-s} Q_{\mathfrak{t}(\ell)}(\alpha_\ell) = \prod_{\ell=1}^{2K} \phi(\alpha_{\tau(\ell)})^{-s} Q_{\mathfrak{t}(\tau(\ell))}(\alpha_{\tau(\ell)}),$$

and hence (5.22) becomes

$$\frac{1}{2^K K!} \sum_{\tau \in S_{2K}} \mathrm{sgn}(\tau) \int_{\mathbb{R}^{2K}} \left\{ \prod_{k=1}^{K} \phi(\alpha_{\tau(2k-1)})^{-s} \phi(\alpha_{\tau(2k)})^{-s} \, \mathrm{sgn}(\alpha_{\tau(2k)} - \alpha_{\tau(2k-1)}) \right.$$
$$\left. \times \, Q_{\mathfrak{t}\circ\tau(2k-1)}(\alpha_{\tau(2k-1)}) Q_{\mathfrak{t}\circ\tau(2k)}(\alpha_{\tau(2k)}) \right\} d\lambda_L(\boldsymbol{\alpha}). \tag{5.23}$$

This integral converges when $\Re(s) > N$, and is written in a form which allows us to employ Fubini's Theorem. Thus (5.23) becomes

$$\frac{1}{2^K K!} \sum_{\tau \in S_{2K}} \operatorname{sgn}(\tau) \prod_{k=1}^{K} \int_{\mathbb{R}^2} \phi(x)^{-s} \phi(y)^{-s} \operatorname{sgn}(y-x)\; Q_{\mathfrak{t}\circ\tau(2k-1)}(x) Q_{\mathfrak{t}\circ\tau(2k)}(y)\, dx\, dy.$$

But this is exactly $\operatorname{Pf}(R_{\mathfrak{t}})$. □

5.4.6 The Proof of Theorem 5.2

The expression for $F_N(\Phi; s)$ given in Equation 5.13 allows us to write $F_N(\Phi; s)$ as

$$\sum_{L+2M=N} \frac{1}{L!M!} \int_{\mathbb{R}^L} \int_{\mathbb{C}^M} \prod_{l=1}^{L} \phi(\alpha_l)^{-s} \prod_{m=1}^{M} \phi(\overline{\beta_m})^{-s} \phi(\beta_m)^{-s} |V(\boldsymbol{\alpha}, \boldsymbol{\beta})|\, d\lambda_{2N}(\boldsymbol{\beta})\, d\lambda_L(\boldsymbol{\alpha}).$$

Substituting the expression for $|V(\boldsymbol{\alpha}, \boldsymbol{\beta})|$ given in Corollary 5.19 we see

$$\begin{aligned} F_N(\Phi; s) = \sum_{L+2M=N} \sum_{\mathfrak{t} \in P_L^N} \operatorname{sgn}(\mathfrak{t}) &\left\{ \frac{1}{L!} \int_{\mathbb{R}^L} \det(A^{\boldsymbol{\alpha}}_{\mathfrak{i},\mathfrak{t}}) \operatorname{Pf}(T^{\boldsymbol{\alpha}}) \prod_{\ell=1}^{L} \phi(\alpha_\ell)^{-s}\, d\lambda_L(\boldsymbol{\alpha}) \right\} \\ &\times \left\{ \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det(B^{\boldsymbol{\beta}}_{\mathfrak{i}',\mathfrak{t}'}) \prod_{m=1}^{M} \phi(\overline{\beta_m})^{-s} \phi(\beta_m)^{-s} \operatorname{sgn}(\Im(\beta_m))\, d\lambda_{2M}(\boldsymbol{\beta}) \right\}. \end{aligned}$$

But, by Lemma 5.21 and Lemma 5.20, we may replace the expressions in braces with $\operatorname{Pf}(R_{\mathfrak{t}})$ and $\operatorname{Pf}(C_{\mathfrak{t}'})$ (resp.). Thus,

$$F_N(\Phi; s) = \sum_{L+2M=N} \sum_{\mathfrak{t} \in P_L^N} \operatorname{sgn}(\mathfrak{t}) \operatorname{Pf}(R_{\mathfrak{t}}) \operatorname{Pf}(C_{\mathfrak{t}'}).$$

Now, since N is even, the first sum may be reindexed by setting $K = 2L$, and $N = 2J$. In which case,

$$F_N(\Phi; s) = \sum_{K=0}^{J} \sum_{\mathfrak{t} \in P_{2K}^{2J}} \operatorname{sgn}(\mathfrak{t}) \operatorname{Pf}(R_{\mathfrak{t}}) \operatorname{Pf}(C_{\mathfrak{t}'}).$$

Recalling that $R = R_Q$ and $C = C_Q$, Lemma 5.14 shows

$$F_N(\Phi; s) = \operatorname{Pf}(R_Q + C_Q),$$

and Theorem 5.2 is proved.

5.4.7 The Proof of Theorem 5.3

Let Φ is a multiplicative distance function such that $\phi(-\beta) = \phi(\beta)$ and $\phi(\overline{\beta}) = \phi(\beta)$. We will first show that the Theorem is true when $Q = \{1, \gamma, \gamma^2, \ldots, \gamma^{N-1}\}$. To do this we will show that $\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{R}} = \langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{C}} = 0$ whenever j and k are positive integers such that $(j - k) \equiv 0 \bmod 2$. This will allow us to use Lemma 5.15 to write $\mathrm{Pf}(U_Q)$ as a determinant. Firstly,

$$\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{R}} = \int_{\infty}^{\infty} \int_{\infty}^{\infty} \phi(x)^{-s} \phi(y)^{-s} x^{j-1} y^{k-1} \operatorname{sgn}(y - x)\, dx\, dy,$$

from which the change of variables $x \mapsto -x, y \mapsto -y$ shows that

$$\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{R}} = (-1)^{j+k} \langle \gamma^{k-1}, \gamma^{j-1} \rangle_{\mathbb{R}}.$$

But, since this is a skew-symmetric inner product we must have $\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{R}} = 0$ if $(j - k) \equiv 0 \bmod 2$. Similarly,

$$\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{C}} = \int_{\mathbb{C}} \phi(\overline{\beta})^{-s} \phi(\beta)^{-s} \overline{\beta}^{j-1} \beta^{k-1} \operatorname{sgn}(\Im(\beta))\, d\lambda_2 \beta.$$

But the change of variables $\beta \mapsto -\overline{\beta}$ shows that

$$\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{C}} = (-1)^{j+k} \langle \gamma^{k-1}, \gamma^{j-1} \rangle_{\mathbb{C}}.$$

And hence $\langle \gamma^{j-1}, \gamma^{k-1} \rangle_{\mathbb{C}} = 0$ if $(j - k) \equiv 0 \bmod 2$. Lemma 5.15 reveals that

$$F_N(\Phi; s) = \mathrm{Pf}(U_Q) = \det(A_Q),$$

where $A_Q(j,k) = U_Q(2j-1, 2k) = \langle \gamma^{2j-2}, \gamma^{2k-1} \rangle$.

To see that the theorem is true more generally, let Q' be another family of monic polynomials satisfying the hypotheses of the theorem. Then, by the bilinearity of the skew symmetric inner product, $\langle Q'_j, Q'_k \rangle = 0$ if $(j - k) \equiv 0 \bmod 2$. Thus, we may employ Lemma 5.15 to see that

$$F_N(\Phi; s) = \mathrm{Pf}(U_{Q'}) = \det(A_{Q'}),$$

where $A_{Q'}(j,k) = U_{Q'}(2j-1, 2k) = \langle Q'_{2j-1}, Q'_{2k} \rangle$.

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Vita

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